Journal of Reliability and Statistical Studies (ISSN: 0974-8024)
Vol. 2, Issue 2(2009): 1-10

INFERENCE IN THE MULTIVARIATE EXPONENTIAL MODELS

David D. Hanagal
Department of Statistics, University of Pune, Pune-411007, India.
E Mail: ddh@stats.unipune.ernet.in; david_hanagal@yahoo.co.in

Abstract

Block (1975) extended bivariate exponential distributions (BVEDs) of Freund (1961) and Proschan and Sullo (1974) to multivariate case and called them as Generalized Freund-Weinman's multivariate exponential distributions (MVEDs). In this paper, we obtain MLEs of the parameters and large sample test for testing independence and symmetry of k components in the generalized Freund-Weinman's MVEDs.

Key Words: Fisher information, Generalized likelihood ratio test, Maximum likelihood estimator, Multivariate exponential model, Simultaneous failures.

1. Introduction

Multivariate exponential models can be viewed in the context of failure time distribution of k components. Kotz, Balakrishna and Johnson (2000) discussed seven multivariate exponential distributions. These are generalizations of Freund's BVED by Weinman (1966), the multivariate exponential ($MVED_1$) of Marshall and Olkin (1967), MVED of Block (1974), the MVED of Al-Saadi and Young (1982) which is generalization of BVED of Moran (1967) and Downton (1970), the MVED of Raftery (1984) and O'Cinneide and Reftery (1989), the MVED of Olkin and Tong (1994) and the multivariate exponential obtained by specializing a particular multivariate gamma distribution. The Weinman distribution is a generalization of Freund's distribution but is not a completely satisfactory generalization, since it is restricted to identical marginals which corresponds to symmetry of the marginal life time distribution of k components. Block (1975) generalized the Weinman model to non-identical marginals which leads to a multivariate exponential distribution ($MVED_2$) and it depends on $k2^{k-1}$ parameters. Block (1975) also extended BVED of Block and Basu (1974) to multivariate case which we call as $MVED_3$ model. The fourth model is multivariate extension of Proschan and Sullo's (1974) BVED, which we call as $MVED_4$ and is the combination of both $MVED_1$ and $MVED_2$ models.

The problem of test of independence in the symmetric $MVED_3$ of Block (1975) was studied by Weier and Basu (1980). Their work is based on generalized likelihood ratio test (GLRT) of Barlow et al (1972) who considered the GLRT for testing equality of scale parameters in the ordered gamma distributions, using isotonic regression estimates. Tests of independence as well as symmetry in $MVED_1$ and $MVED_3$ have been studied in detail by Hanagal (1991a, 1993a, 1993b). In this paper, we consider the above problems in $MVED_2$ and $MVED_4$ models. [See related results in Hanagal (1991b)].

In Section 2, we study the inter-relationship between the multivariate exponential models. In Section 3, we obtain MLEs and their asymptotic distribution in these two models. We also determine the confidence intervals of the parameters in these two models. In Section 4 and 5, we develop test of independence and symmetry in these two models.

2. Models for MVED

The p.d.f. of $(X_1,X_2,\ldots,X_k)$ of $MVED_2$ of Block (1975) is given by

$$f(\underline{x}) = (\prod_{j=1}^{k}\theta_{i_j}^{(j-1)})\exp\{-\sum_{j=1}^{k}(\sum_{r=j}^{k}\theta_{i_r}^{(j-1)})(x_{(j)}-x_{(j-1)})\}, 0 = x_{i_0} < x_{i_1} < \ldots < x_{i_k} \tag{2.1}$$

where $x_{(i)}$ = i-th ordered failure time, i.e., i-th minimum of $(X_1,..,X_k)$, I = 1,…,k, $\theta_{i_j}^{(j-1)} = \theta_{i_j|i_1,\ldots,i_{j-1}}^{(j-1)} \geq 0$, j = 1,…,k; $i_1 \neq \ldots \neq i_k = 1,\ldots k$, $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j-1)}$ is failure rate of the life time of the component C_{i_j} when (j-1) components $C_{i_1},\ldots,C_{i_{j-1}}$ failed and $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j-1)}$ is independent of the order of failure of the components $C_{i_1},\ldots,C_{i_{j-1}}$. (See Block (1975) for more details). The double subscripts $i_1 \neq \ldots \neq i_k = 1,\ldots k$ are used because we have k! different regions in the p.d.f $f(\underline{x})$.

In the above model $MVED_2$ of Block (1975), the j-th failure is independent of order of failure of previous (j-1) components and so we have $k2^{k-1}$ parameters in all as can be seen from the following argument. For fixed j, the number of parameters in $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j-1)}$ $i_1 \neq \ldots \neq i_j = 1,\ldots k$ are $k\binom{k-1}{j-1}$. So, for j = 1,2,…,k, the total number of parameters in $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j-1)}$ $i_1 \neq \ldots \neq i_j = 1,\ldots k$ will be $\sum_{j=1}^{k} k\binom{k-1}{j-1} = k2^{k-1}$.

The above model $MVED_2$ has loss of memory property (LMP) but the marginals are not exponentials and are weighted combinations of exponentials.

The random variables $(X_1,..,X_k)$ would be independent if and only if $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j)}$ = $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j-1)}$, j=1,…,k-1; $i_1 \neq \ldots \neq i_k = 1,\ldots k$. Here the hypothesis of independence is a $k(2^{k-1}-1)$ dimensional one and can be shown as follows.

Observe that for fixed i_k and j, the number of parameters in $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j)}$, $i_1 \neq \ldots \neq i_j = 1,\ldots k$ is $\binom{k-1}{j}$. So the total number of parameters in $\theta_{i_j|i_1,\ldots,i_{j-1}}^{(j)}$, j = 1,…,k - 1; $i_1 \neq \ldots \neq i_j = 1,\ldots k$ would become $\sum_{j=1}^{k-1}\binom{k-1}{j} = (2^{k-1}-1)$. These $(2^{k-1}-1)$ parameters all equal to $\theta_{i_k}^{(0)}$ for fixed i_k. So, the hypothesis of independence is a $k(2^{k-1}-1)$ dimensional one.

The symmetry of k components implies identical marginals of all k components. The random variables $(X_1,..,X_k)$ are identically distributed if and only if $\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}} = \theta^{(j-1)}_{i_l|i_1,...,i_{j-1}}$, $j \neq l = 1,...,k;\ i_1 \neq ... \neq i_k = 1,...k$. In this case also the hypothesis of symmetry is a $k(2^{k-1}-1)$ dimensional one and can be shown as follows.

For fixed j, the number of parameters in $\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}}, i_1 \neq ... \neq i_j = 1,...k$ will be $k\binom{k-1}{j-1}-1$ dimensional one. Therefore for j = 1,…, k, the hypothesis of symmetry is a $\sum_{j=1}^{k}\left\{k\binom{k-1}{j-1}-1\right\} = k(2^{k-1}-1)$ dimensional one.

We next derive the pdf of the fourth model $MVED_4$ which is the multivariate extension of BVED of Proschan and Sullo (1974) in the following manner as suggested by Block (1975).

Initially the lifetimes $(X_1,..,X_k)$ follow (k+1) parameters version of $MVED_1$ of Marshall-Olkin (1967) as stated in Proschan-Sullo (1976) with survival function

$$F(\underline{x}) = \exp\left\{-\sum_{i=1}^{k}\theta_i^{(0)} x_i \theta_{k+1}\max(x_1,...,x_k)\right\},$$

$$x_i \geq 0, \theta_i^{(0)} \geq -, i = 1,...,k; \theta_{k+1} \geq 0.$$

We assume that if j(j=1,…,k-1) components fail (and not been replaced) then the survival function of the remaining (k-j) components is

$$\overline{F}(x_{i_{j+1}},...,x_{i_k}) = \exp\left\{-\sum_{r=j+1}^{k}\theta^{(r-1)}_{i_r|i_1,...,i_{r-1}} x_{i_r} - \theta_{k+1}\max(x_{i_{j+1}},...,x_{i_k})\right\},$$

$$i_{j+1} \neq ... \neq i_k = 1,...,k.$$

Incorporating the above modification, we derive the p.d.f. of $MVED_4$ which is given by

$$f(\underline{x}) = \left(\prod_{j=1}^{k-1}\theta_{i_j}^{(j-1)}\right)(\theta_{i_k}^{(k-1)} + \theta_{k+1})\exp\left\{-\sum_{j=1}^{k}\sum_{r=j}^{k}\theta_{i_r}^{(j-1)}(x_{(j)} - x_{(j-1)}) - \theta_{k+1}x_{(k)}\right\},$$

$0 = x_{i_0} < x_{i_1} < ... < x_{i_k}$ w.r.t. Lebesque measure in R_k

$$= \left(\prod_{j=1}^{k-1}\theta_{i_j}^{(j-1)}\right)\theta_{k+1}\exp\left\{-\sum_{j=1}^{k-l+1}\sum_{r=j}^{k}\theta_{i_r}^{(j-1)}(x_{(j)} - x_{(j-1)}) - \theta_{k+1}x_{(k-l+1)}\right\},$$

$0 = x_{i_0} < x_{i_1} < ... < x_{i_{k-l}} < (x_{i_{k-l+1}} = ... = x_{i_k})$ w.r.t. Lebesque measure in R_{k-l+1}, i = 1,…,k; (2.2)

where $x_{(i)}$ = i-th ordered failure time, i = 1,…,k, $\theta_{i_j}^{(j-1)} = \theta^{(j-1)}_{i_j|i_1,...,i_{j-1}} \geq 0$, j = 1,…,k; $i_1 \neq ... \neq i_k = 1,...k$.

In the above model $MVED_4$ also we assume that the j-th failure is independent of the order of failure of previous (j-1) components and so we have $k2^{k-1}+1$ parameters in all in $MVED_4$ model.

When $\theta_{k+1}=0,$ i.e., the probability of simultaneous failures of the components is zero, the $MVED_4$ reduces to $MVED_2$ of Block (1975), which in turn reduces to $MVED_3$ from relation (5.4) of Block (1975). If $\theta^{(j)}_{i_j|i_1,\ldots,i_{j-1}}$ = $\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}$, j=1,…,k-1; $i_1\neq\ldots\neq i_k=1,\ldots k$ i.e., failure of a component doesnot change the parameter of the life distribution of other components, $MVED_4$ reduces to $MVED_1$ of Marshall-Olkin (1967). The random variables $(X_1,..,X_k)$ of MVED4 are independent if and only if $\theta_{k+1}=0$ and $\theta^{(j)}_{i_j|i_1,\ldots,i_{j-1}}$ = $\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}$, j=1,…,k-1; $i_1\neq\ldots\neq i_k=1,\ldots k.$ Here the hypothesis of independence is $k(2^{k-1}-1)+1$ dimensional one.

The random variables $(X_1,..,X_k)$ are identically distributed if and only if $\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}$ =$\theta^{(j-1)}_{i_l|i_1,\ldots,i_{j-1}}$, $j\neq l=1,\ldots,k;\ i_1\neq\ldots\neq i_k=1,\ldots k.$ Here the hypothesis for symmetry is $k(2^{k-1}-1)$ dimensional one.

3. MLEs of the Parameters and their Asymptotic Distributions

We first consider the method of maximum likelihood in $MVED_2$ of Block (1975). Let $\{(x_{il})\}, i=1,\ldots,k; l=1,\ldots,n$ be i.i.d. sample of size n. The likelihood of the sample of size n in $MVED_2$ model is given by

$$L=(\prod_{j=1}^{k}\theta^{(j-1)}_{i_j})^{m^{(j-1)}_{i_j}}\ \exp\{-\sum_{j=1}^{k}(\sum_{r=j}^{k}\theta^{(j-1)}_{i_r})\sum_{l=1}^{n}(x_{(j)l}-x_{(j-1)l})\} \tag{3.1}$$

where $m^{(j-1)}_{i_j}=m^{(j-1)}_{i_j|i_1,..,i_{j-1}}$ = the number of observations when the component C_{i_j} fails after the failure of (j-1) components $C_{i_1},\ldots,C_{i_{j-1}}$ and the failure of the component C_{i_j} is independent of the order of failure of the components $C_{i_1},\ldots,C_{i_{j-1}}$ and $x_{(j)l}$ = the j-th minimum of $(x_{1l},\ldots,x_{kl}), l=1,\ldots,n.$

The expected values of $m^{(j-1)}_{i_j|i_1,..,i_{j-1}}, j=1,\ldots,k$ can be obtained from the p.d.f. of $MVED_2$ which are given by

$$E(m^{(j-1)}_{i_j|i_1,..,i_{j-1}})=nP(\max(X_{i_1},\ldots,X_{i_{j-1}})<X_{i_j}M\min(X_{i_{j+1}},\ldots,X_{i_k})),$$

j = 1,…,k; $i_1\neq\ldots\neq i_k=1,\ldots k$. (3.2)

Here $E(m^{(j-1)}_{i_j|i_1,..,i_{j-1}})$ depends on the parameters $\theta^{(j-1)}_{i_l|i_1,\ldots,i_{j-1}}, j\neq l=1,\ldots,k;$ $i_1\neq\ldots\neq i_k=1,\ldots k.$ The exact expression of $E(m^{(j-1)}_{i_j|i_1,..,i_{j-1}})$ is very difficult to obtain.

The likelihood equations are given by

$$\frac{\partial \log L}{\partial \theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}} = 0 = \frac{m^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}}{\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}} - \sum_{i=1}^{n}(x_{(j)l} - x_{(j-1)l}), \; j = 1,\ldots,k; i_1 \neq \ldots \neq i_k = 1,\ldots,k.$$

Thus the score function w.r.t. the components of the parameters depends only on that component and the components are separable in the sense of Hanagal and Kale (1992).

The MLEs of $\theta^{(j-1)}_{i_l|i_1,\ldots,i_{j-1}}, j \neq l = 1,\ldots,k;\; i_1 \neq \ldots \neq i_k = 1,\ldots k$ are given by

$$\hat{\theta}^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}} = \frac{m^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}}{\sum_{l=1}^{n}(x_{(j)l} - x_{(j-1)l})}, \; j = 1,\ldots,k; i_1 \neq \ldots \neq i_k = 1,\ldots,k.$$

Using $E(m^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}})$ from (3.2), we obtain Fisher information matrix which is diagonal with diagonal elements given by

$$\frac{E[m^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}]}{[\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}]^2}, \; j = 1,\ldots,k; i_1 \neq \ldots \neq i_k = 1,\ldots,k.$$

The parameters $\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}, j \neq l = 1,\ldots,k;\quad i_1 \neq \ldots \neq i_k = 1,\ldots k$ are thus orthogonal. The above Fisher information matrix is positive definite. Here one can very easily check that MLEs satisfy all regularity conditions for consistent asymptotically normal (CAN) estimators. [See Rao (1973) p.347, 364]. Thus using multivariate central limit theorem (MCLT), the MLEs are asymptotically multivariate normal (AMVN) with variance covariance matrix which is diagonal with diagonal elements given by

$$V(\hat{\theta}^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}) = \frac{[\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}]^2}{E[m^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}]}, \; j = 1,\ldots,k; i_1 \neq \ldots \neq i_k = 1,\ldots,k.$$

Hence all the $k2^{k-1}$ MLEs are asymptotically independent. The $100(1-\alpha)\%$ confidence interval for the parameters $\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}}, j \neq l = 1,\ldots,k;\quad i_1 \neq \ldots \neq i_k = 1,\ldots k$ based on the MLEs and their asymptotic variances are given by

$$\hat{\theta}^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}} \pm \xi_{1-\alpha/2}[V(\hat{\theta}^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}})/n]^{1/2}, \; j = 1,\ldots,k; i_1 \neq \ldots \neq i_k = 1,\ldots,k$$

where $\xi_{1-\alpha/2}$ is $100(1-\alpha/2)\%$ point of standard normal variate.

We next consider the method of maximum likelihood in MVED$_4$ model. Letting $\eta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}} = (\theta^{(j-1)}_{i_j|i_1,\ldots,i_{j-1}} + \theta_{k+1}),$ the likelihood of the sample of size n is in MVED$_4$ model is given by

$$L = (\prod_{j=1}^{k}\theta^{(j-1)}_{i_j})^{m^{(j-1)}_{i_j}} (\eta^{(k-1)}_{i_k})^{m^{(k-1)}_{i_k}} \theta_{k+1} \exp\{-\sum_{j=1}^{k}(\sum_{r=j}^{k}\theta^{(j-1)}_{i_r})\sum_{l=1}^{n}(x_{(j)l} - x_{(j-1)l})$$

$$-\eta^{(k-1)}_{i_k}\sum_{l=1}^{n}(x_{(k)l} - x_{(k-1)l}) - \theta_{k+1}\sum_{l=1}^{n}x_{(k-1)l}\}$$

where m_{k+1} is the number of observations with at least two simultaneous failures of k components and $m_{i_j}^{(j-1)} = m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)}$, is as defined in the likelihood of MVED$_2$ from expression (3.1).

The expected values of m_{k+1} and $m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)}, j = 1,\dots,k$ can be obtained from the p.d.f. of MVED$_4$ which are given by

$$E(m_{k+1}) = nP[\max(X_{i_1},\dots,X_{i_{k-r-1}}) < X_{i_r} = \dots = X_{i_k} = X_{(k)}]$$

r = 1,…, k-1; $i_1 \neq \dots \neq i_k = 1,\dots k$ and $E(m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)})$ as defined in (3.2). Here also $E(m_{k+1})$ and $E(m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)})$ depends on the parameters $\theta_{i_l|i_1,\dots,i_{j-1}}^{(j-1)}, j \neq l = 1,\dots,k;$ $\eta_{i_k|i_1,\dots,i_{k-1}}^{(k-1)}, i_1 \neq \dots \neq i_k = 1,\dots k$ and θ_{k+1}. The exact expression of $E(m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)})$ is very difficult to obtain.

The likelihood equations are given by

$$\frac{\partial \log L}{\partial \theta_{i_j|i_1,\dots,i_{j-1}}^{(j-1)}} = 0 = \frac{m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)}}{\theta_{i_j|i_1,\dots,i_{j-1}}^{(j-1)}} - \sum_{i=1}^{n} (x_{(j)l} - x_{(j-1)l}),\ j = 1,\dots,k; i_1 \neq \dots \neq i_k = 1,\dots,k$$

$$\frac{\partial \log L}{\partial \eta_{i_k|i_1,\dots,i_{k-1}}^{(k-1)}} = 0 = \frac{m_{i_k|i_1,\dots,i_{k-1}}^{(k-1)}}{\eta_{i_k|i_1,\dots,i_{k-1}}^{(k-1)}} - \sum_{i=1}^{n} (x_{(k)l} - x_{(k-1)l}), i_k = 1,\dots,k$$

$$\frac{\partial \log L}{\partial \theta_{k+1}} = 0 = \frac{m_{k+1}}{\theta_{k+1}} - \sum_{i=1}^{n} x_{(k-1)l}.$$

Note that the parameters in this model are also separable in the sense of Hanagal and Kale (1992). The MLEs of these $k2^{k-1}+1$ parameters are given by

$$\hat{\theta}_{i_j|i_1,\dots,i_{j-1}}^{(j-1)} = \frac{m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)}}{\sum_{l=1}^{n} (x_{(j)l} - x_{(j-1)l})},\ j = 1,\dots,k; i_1 \neq \dots \neq i_k = 1,\dots,k$$

$$\hat{\eta}_{i_k|i_1,\dots,i_{k-1}}^{(k-1)} = \frac{m_{i_k|i_1,\dots,i_{k-1}}^{(k-1)}}{\sum_{l=1}^{n} (x_{(k)l} - x_{(k-1)l})}, i_k = 1,\dots,k$$

$$\hat{\theta}_{k+1} = \frac{m_{k+1}}{\sum_{l=1}^{n} x_{(k-1)l}}.$$

Using $E(m_{i_j|i_1,\dots,i_{j-1}}^{(j-1)})$ and $E(m_{k+1})$, we obtain Fisher information matrix which is diagonal with diagonal elements given by

$$\frac{E[m^{(j-1)}_{i_j|i_1,...,i_{j-1}}]}{[\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}}]^2}, j=1,...,k; i_1 \neq ... \neq i_k = 1,...,k$$

$$\frac{E[m^{(k-1)}_{i_k|i_1,...,i_{k-1}}]}{[\theta^{(j-1)}_{i_k|i_1,...,i_{k-1}}]^2}, i_k = 1,...,k$$

$$\frac{E(m_{k+1})}{\theta^2_{k+1}}$$

The parameters $\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}}, j \neq l = 1,...,k; \eta^{(k-1)}_{i_k|i_1,...,i_{k-1}}, \ i_1 \neq ... \neq i_k = 1,...k$ and θ_{k+1} are thus orthogonal. The above Fisher information matrix is positive definite. Here one can very easily check that MLEs satisfy all regularity conditions for consistent asymptotically normal (CAN) estimators. [See Rao (1973)]. Thus using multivariate central limit theorem (MCLT), the MLEs are asymptotically multivariate normal (AMVN) with variance covariance matrix which is diagonal with diagonal elements given by

$$V(\hat{\theta}^{(j-1)}_{i_j|i_1,...,i_{j-1}}) = \frac{[\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}}]^2}{E[m^{(j-1)}_{i_j|i_1,...,i_{j-1}}]}, j=1,...,k; i_1 \neq ... \neq i_k = 1,...,k$$

$$V(\hat{\eta}^{(j-1)}_{i_j|i_1,...,i_{j-1}}) = \frac{[\eta^{(k-1)}_{i_k|i_1,...,i_{j-1}}]^2}{E[m^{(k-1)}_{i_k|i_1,...,i_{j-1}}]}, i_k = 1,...,k$$

$$V(\theta_{k+1}) = \frac{\theta^2_{k+1}}{E(m_{k+1})}.$$

Hence all the $k2^{k-1}+1$ MLEs are asymptotically independent. We can also obtain MLEs of $\theta^{(k-1)}_{i_k|i_1,...,i_{k-1}}, \ i_1 \neq ... \neq i_k = 1,...k$ by the relation $\hat{\theta}^{(k-1)}_{i_k|i_1,...,i_{k-1}} = \hat{\eta}^{(k-1)}_{i_k|i_1,...,i_{k-1}} - \hat{\theta}_{k+1}$, $i_1 \neq ... \neq i_k = 1,...k$. The asymptotic variances of $\theta^{(k-1)}_{i_k|i_1,...,i_{k-1}}, \ i_1 \neq ... \neq i_k = 1,...k$ can also be obtained from the relation

$$V(\hat{\theta}^{(k-1)}_{i_k|i_1,...,i_{k-1}}) = V(\hat{\eta}^{(k-1)}_{i_k|i_1,...,i_{k-1}}) + V(\hat{\theta}_{k+1}).$$

The $100(1-\alpha)\%$ confidence interval for the parameters $\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}}, j \neq l = 1,...,k; \ i_1 \neq ... \neq i_k = 1,...k$ and θ_{k+1} based on the MLEs and their asymptotic variances are given by

$$\hat{\theta}^{(j-1)}_{i_l|i_1,...,i_{j-1}} \pm \xi_{1-\alpha/2}[V(\hat{\theta}^{(j-1)}_{i_l|i_1,...,i_{j-1}})/n]^{1/2}, j=1,...,k; i_1 \neq ... \neq i_k = 1,...,k$$

$$\hat{\theta}_{k+1} \pm \xi_{1-\alpha/2}[V(\hat{\theta}_{k+1})/n]^{1/2}$$

where $\xi_{1-\alpha/2}$ is $100(1-\alpha/2)\%$ point of standard normal variate.

4. Test for Independence

We first consider the test of independence in $MVED_4$ model. In this model, the hypothesis of independence of k components corresponds to

$H_0 : \theta^{(j)}_{i_k|i_1,...,i_j} = \theta^{(j-1)}_{i_k|i_1,...,i_{j-1}}, j = 1,...,k-1; i_1 \neq ... \neq i_k = 1,...,k$ and $\theta_{k+1} = 0.$ This is $k(2^{k-1}-1)+1$ dimensional one. The test is based on

$W = (\hat{\theta}^{(j)}_{i_k|i_1,...,i_j} - \hat{\theta}^{(j-1)}_{i_k|i_1,...,i_{j-1}}, j = 1,...,k-1; i_1 \neq ... \neq i_k = 1,...,k; \hat{\theta}_{k+1})' = (W_1', \hat{\theta}_{k+1})',$

where W is a vector of order $k(2^{k-1}-1)+1$ and W_1 is a vector of order $k(2^{k-1}-1)$. The exact distribution of W is very difficult to find out but its asymptotic distribution can be obtained using the results of Section 3.

One can obtain GLRT based on $-2[\log \lambda(\underline{x}) = -2[\log L_0(\underline{x}) - \log L_1(\underline{x})].$ An approximation to this test procedure is obtained by using the fact that W is AMVN($\mu, \Sigma / n$) where

$\mu = (\theta^{(j)}_{i_k|i_1,...,i_j} - \theta^{(j-1)}_{i_k|i_1,...,i_{j-1}}, j = 1,...,k-1; i_1 \neq ... \neq i_k = 1,...,k; \theta_{k+1})' = (\mu_1', \theta_{k+1})',$

, μ is a vector of order $k(2^{k-1}-1)+1$ and μ_1 is a vector of order $k(2^{k-1}-1)$ and Σ is the variance-covariance matrix of W. But Σ depends on $k2^{k-1}+1$ unknown parameters, we studentize (estimating the variance-covariance matrix Σ by $\hat{\Sigma}$ from the MLEs under $H_0 \cup H_1$) and construct the test statistic $nW'\hat{\Sigma}^{-1}W$ which is asymptotically chi-square with $k(2^{k-1}-1)+1$ d.f. under H_0. This is well-known as Wald's test. [In the case of GLRT, the variance-covariance matrix (Σ) of W is estimated by MLEs under H_0]. We reject H_0 if $nW'\hat{\Sigma}^{-1}W > \chi^2_{p+1,1-\alpha}$ where $\chi^2_{p+1,1-\alpha}$ is $100(1-\alpha)\%$ point of chi-square variate, $p = k(2^{k-1}-1)$. The power function increases monotonically with non-centrality parameter $n\mu'\Sigma^{-1}\mu.$

We next consider $MVED_2$ model. The hypothesis of independence corresponds to $H_0 : \mu_1 = 0$ versus $H_1 : \mu_1 \neq 0$ where

$\mu_1 = (\theta^{(j)}_{i_k|i_1,...,i_j} - \theta^{(j-1)}_{i_k|i_1,...,i_{j-1}}, j = 1,...,k-1; i_1 \neq ... \neq i_k = 1,...,k)'.$ The hypothesis is $k(2^{k-1}-1)$ dimensional one. Here the test statistic is $nW_1'\hat{\Sigma}_1^{-1}W_1$ which is asymptotically chi-square with $p = k(2^{k-1}-1)$ d.f. under H_0 where Σ_1 is the variance-covariance matrix W_1 and is estimated by $\hat{\Sigma}_1$ from the MLEs. For the alternative $H_1 : \mu_1 \neq 0$, we reject H_0 if $nW_1'\hat{\Sigma}_1^{-1}W_1 > \chi^2_{p,1-\alpha}$. The power function increases monotonically with non-centrality parameter $n\mu_1'\Sigma_1^{-1}\mu_1.$

5. Test for Symmetry

We consider the test for symmetry in both $MVED_2$ and $MVED_4$ models where the hypothesis of symmetry is $k(2^{k-1}-1)$ dimensional one i.e., $H_0 : \mu_2 = 0$ versus $H_1 : \mu_2 \neq 0$ where

$\mu_2 = (\theta^{(j-1)}_{i_j|i_1,...,i_{j-1}} - \theta^{(j-1)}_{i_l|i_1,...,i_{j-1}}, j \neq l = 1,...,k-1; i_1 \neq ... \neq i_k = 1,...,k)'$ in both MVED$_2$ and MVED$_4$ models where μ_2 is a vector of order $p = k(2^{k-1} - 1)$. The test statistic is based on

$W_2 = (\hat{\theta}^{(j-1)}_{i_j|i_1,...,i_{j-1}} - \hat{\theta}^{(j-1)}_{i_l|i_1,...,i_{j-1}}', j \neq l = 1,...,k; i_1 \neq ... \neq i_k = 1,...,k)'$ where W_2 is a vector of order p. The exact distribution of W_2 is very difficult to find out but it s asymptotic distribution can be obtained using the results of Section 3.

One can obtain GLRT based on $-2\log\lambda(\underline{x})$. An approximation to this test procedure is obtained by using the result that W_2 is AMVN($\mu_2, \Sigma_2 / n$) where Σ_2 is the variance-covariance matrix of W_2. But Σ_2 depends on unknown parameters, we studentize (estimating the variance-covariance matrix Σ_2 by $\hat{\Sigma}_2$ from the MLEs under $H_0 \cup H_1$) and construct the test statistic $nW_2'\hat{\Sigma}_2^{-1}W_2$ which is asymptotically chi-square with $k(2^{k-1} - 1)$ d.f. under H_0. For the alternative $H_1 : \mu_2 \neq 0$, we reject H$_0$ if $nW_2'\hat{\Sigma}_2^{-1}W_2 > \chi^2_{p,1-\alpha}$. The power function increases monotonically with non-centrality parameter $n\mu_2'\Sigma_2^{-1}\mu_2$.

Numerical Study

The numerical study of estimation of the parameters and testing for independence and symmetry of MVED$_4$ when k = 2 have been done by Hanagal (1992). MVED$_2$ is sub-model of MVED$_4$ and all the estimation and testing procedures of MVED$_2$ can be obtained from MVED$_4$ model by substituting $\theta_{k+1} = 0$.

Acknowledgment

I thank the referees for the suggestions and comments.

References

1. Al-Saadi, S.D. and Young, D.H. (1982). A test for independence in a multivariate exponential distribution with equal correlation coefficient. Journal of Statistical Computation and Simulation, 14, 219-27.
2. Barlow, R.E., Barthlomew, D.J., Bremner, J.M. and Brunk, H.D. (1972). Statistical Interface Under Order Restriction. New York, John Wiley & Sons.
3. Block, H.W. and Basu, A.P. (1974). A continuous bivariate exponential extension. Journal of the American Statistical Association, 69, 1031-37.
4. Block, H.W. (1975). Continuous multivariate exponential extensions. Reliabilty and Fault Analysis, Eds. R.E. Barlow, J.B. Fussel and N.D. Singpurwala. Philadelphia; SIAM, 285-306.
5. Downton, F. (1970). Bivariate exponential distributions in reliability theory. Journal of the Royal Statistical Society, Series B, 32, 408-17.
6. Freund, J.E. (1961). Bivariate extension of exponential distribution. Journal of the American Statistical Association, 56, 971-77.
7. Hanagal, D.D. (1991a). Large sample tests of independence and symmetry in multivariate exponential distribution. Journal of the Indian statistical Association, 29(2), 89-93.
8. Hanagal, D.D. (1991b). Some Contributions to the Inference in Bivariate and Multivariate Exponential Distributions. Ph.D. thesis, University of Pune, India.
9. Hanagal, D.D. (1992). Some inference results in modified Freund's bivariate exponential distribution. Biometrical Journal, 34(6), 745-56.

10. Hanagal, D.D. (1993a). Some inference results in an absolutely continuous multivariate exponential model of Block. Statistics and Probability Letters, 16(3), 177-80.
11. Hanagal, D. D. (1993b). Some inference results in several symmetric multivariate exponential models. Communications in Statistics, Theory and Methods, 22(9), 2549-66.
12. Hanagal, D.D. and Kale, B.K. (1992). Large sample tests for testing symmetry and independence in some bivariate exponential models. Communications in Statistics, Theory and Methods, 21, 2625-43.
13. Kotz, S., Balakrishnan, N. and Johnson, N.L. (2000). Continuous Multivariate Distributions: Models and Applications. Second Edition. John Wiley & Sons.
14. Marshall, A.W. and Olkin, I. 91967). Multivariate exponential distribution. Journal of the American Statistical Association, 62, 32-44.
15. Moran, P.A.P. (1967). Testing for correlation between non-negative varieties. Biometrka, 54, 385-94.
16. O'Cinneide, C.A. and Raftery, A.E. (1989). Continuous multivariate exponential distribution that is multivariate phase type. Statistics & Probability Letters, 7, 323-25.
17. Olkin, I. and Tong, Y.L. (1994). Positive dependence of a class of multivariate exponential distributions, SIAM Journal on Control and Optimization, 32, 965-74.
18. Proschan, F. and Sullo, P. (1974). Estimating the parameters of the bivariate exponential distributions in several sampling situations. Reliability and Biometry. Eds. F. Proschan and R.J. Surfling, Philadelphia: SIAM, 423-40.
19. Proschan, F. and Sullo, P. (1976). Estimating the parameters of the multivariate exponential distribution. Journal of the American Statistical Association, 71, 465-72.
20. Raftery, A.E. (1984). A continuous multivariate exponential distribution. Communications in Statistics, Theory and Methods, 13, 947-65.
21. Rao, C.R. (1973). Linear Statistical Inference and its Applications. Wiley Eastern Limited.
22. Weier, D.R. and Basu, A. P. (1980). Testing for independence in multivariate exponential distributions. Australian journal of Statistics, 22(3), 276-88.
23. Weinman, D.G. (1966). Multivariate extension of the exponential distribution. Ph.D. thesis, Arizona State University, USA.