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SOME IMPORTANT STATISTICAL PROPERTIES, INFORMATION MEASURES AND ESTIMATIONS OF SIZE BIASED GENERALIZED GAMMA DISTRIBUTION

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Abstract
In this paper, a new class of Size-biased Generalized Gamma (SBGG) distribution is defined. A Size-biased Generalized Gamma (SBGG) distribution, a particular case of weighted Generalized Gamma distribution, taking the weights as the variate values has been defined. The important statistical properties including hazard functions, reverse hazard functions, mode, moment generating function, characteristic function, Shannon's entropy, generalized entropy and Fisher's information matrix of the new model have been derived and studied. Here, we also study SBGG entropy estimation, Akaike and Bayesian information criterion. A likelihood ratio test for size-biasedness is conducted. The estimation of parameters is obtained by employing the classical methods of estimation especially method of moments and maximum likelihood estimator.

Key Words: Size Biased Generalized Gamma Distribution, Shannon's Entropy, Generalized Entropy, Fisher's Information Matrix, Likelihood Ratio Test, Maximum Likelihood Estimator.

1. Introduction
The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to measure of the unit size, resulting distribution is called size-biased. Size biased distributions are a special case of the more general form known as weighted distributions. These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non –experimental, non–replicated and non –random categories. Prentice (1974) resolved the convergence problem using a nonlinear transformation of Generalized Gamma model. However, despite its long history and growing use in various applications, the Generalized Gamma family and its properties has-been remarkably presented in different papers. Hwang *et al* (2006) introduced a new moment estimation of parameters of the generalized gamma distribution using it's characterization. The Size biased Generalized Gamma (SBGG) distribution presents a flexible family in the varieties of shapes and hazard functions for modelling duration. It

was introduced by Ahmed et.al (2013a). The SBGG family, which encompasses exponential $(\beta=1,\ k=0)$ and size biased exponential $(\beta=k=1)$ Mir *et.al* (2013) as a subfamilies, and Size biased Gamma distribution $(\beta=1)$ Ahmed *et.al* (2013b) as a particular case is introduced. The Probability Density function of the Size biased Generalized Gamma distribution is given by:

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\, e^{-(\lambda x)^{\beta}} \tag{1.1}$$

For $\lambda>0, k\geq 0, \beta>0\ and\ \ k\beta>1$

Cumulative Distribution function (CDF) is given as:

$$F_S(x;\lambda,\beta,k)=\frac{\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} \tag{1.2}$$

Where $\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)$ is an incomplete Size biased Generalized Gamma function.

The m^th^ non-central moment of SBGG is given by

$$E(X^m)=\frac{\Gamma\left(k+\frac{m+1}{\beta}\right)}{\lambda^m\,\Gamma\left(k+\frac{1}{\beta}\right)}\ ;\ m=1,2,........ \tag{1.3}$$

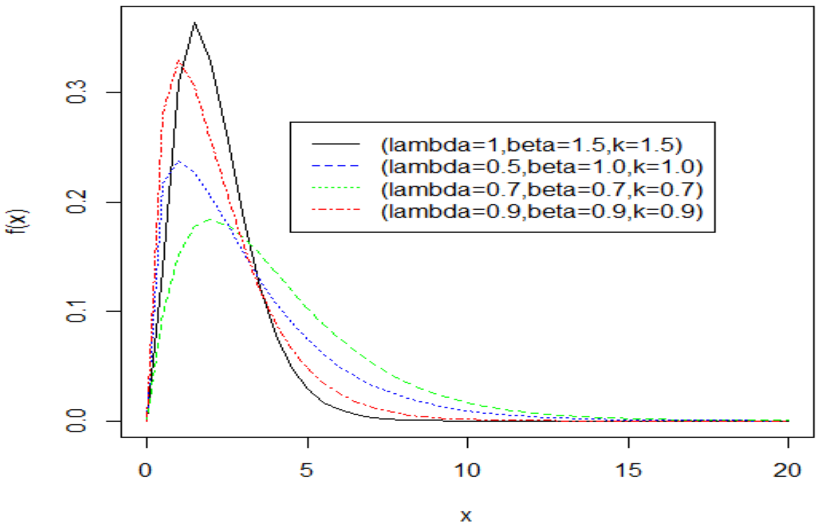
Using the equation (1.3), the mean and variance of the SBGG are given by

$$E(X)\ =\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\,\Gamma\left(k+\frac{1}{\beta}\right)} \tag{1.4}$$

$$V(X)=\frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)-\Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{1}{\beta}\right)} \tag{1.5}$$

The coefficient of variation of SBGG is given by

$$CV = \frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)-\Gamma^2\left(k+\frac{2}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{2}{\beta}\right)} \tag{1.6}$$



Pdf plot for size biased generalized weighted gamma (lambda,beta,k)

From the above figure, we can interpret that the peakedness of a probability density curve increases as we increases the values of the parameters of Size biased Generalized Gamma function.

The moment generating of SBGG distribution is given as:

$$E\left(e^{tx^{\beta}}\right)=\frac{\lambda^{k\beta+1}}{\left(\lambda^{\beta}-t\right)^{k+\frac{1}{\beta}}} \tag{1.7}$$

Substitute $\beta = 1$ in the above relation (1.7), we get the moment generating function of Size biased Gamma Distribution (see Reshi *et. al* (2014)) which is given as:

$$E\left(e^{tx}\right)=\left(\frac{\lambda}{\lambda-t}\right)^{k+1} \tag{1.8}$$

The Characteristic function of SBGG distribution is given as:

$$\Phi_x(t) = \frac{\lambda^{k\beta+1}}{\left(\lambda^{\beta} - it\right)^{k+\frac{1}{\beta}}} \tag{1.9}$$

Substitute $\beta = 1$ in the above relation (1.9), we get Characteristic function of Size biased Gamma Distribution
(see Reshi *et. al* (2014)) which is given as:

$$\Phi_x(t) = \left(\frac{\lambda}{\lambda - it}\right)^{k+1} \tag{1.10}$$

2. Reliability Measures of Size-biased Generalized Gamma Distribution

The hazard function for the Size biased Generalized Gamma distribution is given as:

$$h_s(x;\lambda,\beta,k) = \frac{f_s(x;\lambda,\beta,k)}{1 - F_s(x;\lambda,\beta,k)}$$

$$h_s(x;\lambda,\beta,k) = \frac{\lambda\beta(\lambda x)^{k\beta} e^{-(\lambda x)^{\beta}}}{\Gamma\left(k+\frac{1}{\beta}\right) - \gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)} \tag{2.1}$$

Where $\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)$ is an incomplete Size is biased Generalized Gamma function.

The reverse hazard function for the Size biased Generalized Gamma distribution is given as:

$$h_{rv}(x;\lambda,\beta,k) = \frac{f_s(x;\lambda,\beta,k)}{F_s(x;\lambda,\beta,k)}$$

$$h_{rv}(x;\lambda,\beta,k) = \frac{\lambda\beta(\lambda x)^{k\beta} e^{-(\lambda x)^{\beta}}}{\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)} \tag{2.2}$$

Where $\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)$ is an incomplete Size is biased Generalized Gamma function.

Theorem 2.1:. Define $n(x;\lambda,\beta,k) = -\frac{f_s'(x;\lambda,\beta,k)}{f_s(x,\lambda,\beta,k)}$,where $f'(x;\lambda,\beta,k)$ is the first derivative of $f(x;\lambda,\beta,k)$ with respect to x. Furthermore, suppose that the first derivative of $n(x;\lambda,\beta,k)$ exist.

a) If $n'(x;\lambda,\beta,k)<0,$ for all x > 0, then the hazard function is monotonically decreasing.
b) If $n'(x;\lambda,\beta,k)>0,$ for all x > 0, then the hazard function is monotonically increasing.
c) Suppose there exist x_0 such that $n'(x;\lambda,\beta,k)<0, \text{for all} 0<x<x_0$ and $n'(x;\lambda,\beta,k)>0, \text{for all} 0>x_0$, In addition, $\lim_{x\to 0} f(x)=0$ then the hazard function is upside down bathtub shape.

Proof: Using equation (1.1), the derivative of the $f_s(x;\lambda,\beta,k)$ is given by:

$$f_s'(x;\lambda,\beta,k)=f_s(x;\lambda,\beta,k)\left(\frac{k\beta-\beta(\lambda x)^\beta}{x}\right)$$

Therefore, $n(x;\lambda,\beta,k)=-\frac{f_s'(x;\lambda,\beta,k)}{f_s(x,\lambda,\beta,k)}$

$$n(x;\lambda,\beta,k)=\frac{\beta\left[(\lambda x)^\beta-k\right]}{x}$$ and

$$n'(x;\lambda,\beta,k)=\frac{\beta(\beta-1)(\lambda x)^\beta+k\beta}{x^2} \tag{2.3}$$

a) If $\beta\ge 1$, then $n'(x;\lambda,\beta,k)>0,$ for all x > 0, then the hazard function is monotonically increasing.

b) If $\beta<1$, then $n'(x;\lambda,\beta,k)<0,$ for all x > 0, then the hazard function is monotonically decreasing.

c) Suppose $\beta<1$. Let $x^*=\left[\frac{-k}{(\beta-1)\lambda^\beta}\right]^{\frac{1}{\beta}}$. Then $n'(x;\lambda,\beta,k)=0, \text{ if } x^*=x$

$\text{if } x^*<x, n'(x;\lambda,\beta,k)>0 \text{ and } n'(x;\lambda,\beta,k)<0 \text{ if } x>x^*$. If, $\lim_{x\to 0} f(x)=0$, then the hazard function is upside down bathtub shape.

3. Structural properties and information measures of Size biased Generalized Gamma Distribution

In this section, we derive some structural properties and information measures of Size-biased generalized gamma distribution.

3.1 Mode of Size biased generalized gamma distribution

The probability distribution of Size biased Generalized Gamma distribution can be obtained as:

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}$$

In order to discuss monotonicity of size biased Generalized Gamma distribution, we take the logarithm of its probability density function:

$$\ln f_S(x;\lambda,\beta,k)=\ln\left(\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)+k\beta\ln(\lambda x)-\left(\lambda x\right)^{\beta} \tag{3.1}$$

Where $C=\ln\left(\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)$ is a constant.

Note that $\frac{\partial \ln f(x;\lambda,\beta,k)}{\partial x}=0 \Leftrightarrow x=\frac{k^{\frac{1}{\beta}}}{\lambda}$ and $\frac{\partial^2 \ln f(x;\lambda,\beta,k)}{\partial^2 x}<0$

Therefore, the mode of size biased generalized gamma distribution is given by:

$$x=\frac{k^{\frac{1}{\beta}}}{\lambda} \tag{3.2}$$

3.2 Shannon's entropy of Size-biased Generalized Gamma Distribution

The concept of Shannon's entropy is the central role of information theory, sometimes referred as measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. Henceforth we assume that log is to the base 2 and entropy is expressed in bits. In information theory, thus far a maximum entropy (ME) derivation of GG is found in Kapur (1989). For deriving the entropy of the size-biased Generalized Gamma distribution, we need the two definitions that are more details of them can be found in Shannon (1948).

Theorem.3.1 Let $x_1,x_2,x_3...x_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized gamma density

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}\ ,\ x>0,\lambda>0,\beta>0,k\geq 0$$

Then Shannon's entropy of Size-biased Generalized Gamma Distribution is:

$$H\left(f\left(x;\lambda,k,\beta\right)\right)=-\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)-k\left[\Psi\left(k+\frac{1}{\beta}\right)-\log\lambda^{\beta}\right]+\frac{\Gamma\left(k+\frac{1}{\beta}+1\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}$$

Proof: Shannon's entropy is defined as:

$$H\left[f(x;\alpha,\beta,k)\right]=E\left[-\log\{f(x;\alpha,\beta,k\}\right]$$

$$H\left[f(x;\alpha,\beta,k)\right]=E\left[-\log\left\{\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}x^{k\beta}e^{-(\lambda x)^{\beta}}\right\}\right]$$

$$H\left(f\left(x;\lambda,k,\beta\right)\right)=-\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)-k\beta\, E\left(\log x\right)+\lambda^{\beta}\mathrm{E}\left(x\right)^{\beta}$$

$$H\left(f\left(x;\lambda,k,\beta\right)\right)=-\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)-k\beta\, E\left(\log x\right)+\frac{\Gamma\left(k+\frac{1}{\beta}+1\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} \qquad (3.3)$$

Now, $E(\log(x))=\int_{0}^{\infty}\log x f(x;\alpha,\beta,k)dx$

$$E(\log(x))=\int_{0}^{\infty}\log x\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}x^{k\beta}\,e^{-(\lambda x)^{\beta}}dx \qquad (3.4)$$

$$E(\log(x))=\frac{1}{\beta}\Psi\left(k+\frac{1}{\beta}\right)-\log\lambda \qquad (3.5)$$

Substitute the value of equation (3.5) in equation (3.4), we have

$$H\left(f(x;\lambda,k,\beta)\right) = -\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right) - k\left[\Psi\left(k+\frac{1}{\beta}\right) - \log\lambda^{\beta}\right] + \frac{\Gamma\left(k+\frac{`1}{\beta}+1\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.6}$$

Where $\Psi\left(k+\frac{1}{\beta}\right)$ is a digamma function.

3.3 The Generalized entropy of size-biased Generalized Gamma Distribution

Generalized entropy is often used in econometrics Golan (2006). It is indexed by a single parameter α. The generalized entropy is defined to be $I_\alpha = \frac{v_\alpha u^{-\alpha} - 1}{\alpha(\alpha-1)}; \alpha \neq 0,1$ and $v_\alpha = \int_0^\infty x^\alpha f_s(x;\alpha,\beta,k)dx$

We know that

$$v_\alpha = \frac{\Gamma\left(k+\frac{\alpha+1}{\beta}\right)}{\lambda^\alpha \Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.7}$$

Substituting above values, we get

$$I_\alpha = \frac{\frac{\Gamma\left(k+\frac{\alpha+1}{\beta}\right)}{\lambda^\alpha \Gamma\left(k+\frac{1}{\beta}\right)}\left[\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}\right]^{-\alpha} - 1}{\alpha(\alpha-1)} \tag{3.8}$$

$$I_\alpha = \frac{\Gamma\left(k+\frac{\alpha+1}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)^{\alpha-1} - \Gamma^\alpha\left(k+\frac{2}{\beta}\right)}{\alpha(\alpha-1)}\left[\Gamma^\alpha\left(k+\frac{2}{\beta}\right)\right] \tag{3.9}$$

3.4 Fisher's information matrix of size-biased Generalized Gamma Distribution

The Fisher information is that a random variable ‘X’ contains about the parameter θ is given by

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}\log\left(f(x;\theta)\right)\right]^2$$

Now, if log f(x; θ) is twice differentiable with respect to θ under certain regularity conditions, Fisher's information is given by:

$$I(\theta) = E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log\left(f(x;\theta)\right)\right]$$

The Size biased generalized gamma distribution has a probability density function of the form:

$$f_s(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\, e^{-(\lambda x)^\beta} \tag{3.10}$$

Applying log on both sides in above equation (3.10), we have

$$\log f_s(x;\lambda,\beta,k) = \log\beta + (k\beta+1)\log\lambda - \log\Gamma\left(k+\frac{1}{\beta}\right) - \lambda^\beta x^\beta \tag{3.11}$$

Differentiating equation (3.11) partially with respect to λ, β and k we get

$$\frac{\partial\log f_s(x;\lambda,\beta,k)}{\partial\lambda} = \frac{k\beta - \beta(\lambda x)^\beta + 1}{\lambda} \tag{3.12}$$

$$\frac{\partial\log f_s(x;\lambda,\beta,k)}{\partial\beta} = \frac{\psi\left(k+\frac{1}{\beta}\right)}{\beta^2} + \frac{1}{\beta} + k\log\lambda - (\lambda x)^\beta\log(\lambda x) \tag{3.13}$$

$$\frac{\partial\log f_s(x;\lambda,\beta,k)}{\partial k} = \beta\log\lambda - \psi\left(k+\frac{1}{\beta}\right) \tag{3.14}$$

Differentiating again the above equation partially (3.12), (3.13) and (3.14) with respect to λ, β and k we have

$$\frac{\partial^2\log f_s(x;\lambda,\beta,k)}{\partial\lambda^2} = \frac{-(k\beta+1) - \beta(\beta-1)(\lambda)^\beta x^\beta - 1}{\lambda^2} \tag{3.15}$$

$$\frac{\partial^2\log f_s(x;\lambda,\beta,k)}{\partial\beta\partial\lambda} = \frac{k - \lambda^\beta\left[(\beta\log\lambda+1)x^\beta + \beta x^\beta\log x\right]}{\lambda} \tag{3.16}$$

$$\frac{\partial^2\log f_s(x;\lambda,\beta,k)}{\partial k\partial\lambda} = \frac{\beta}{\lambda} \tag{3.17}$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta^2} = -\left(\begin{array}{l} \frac{1}{\beta^2} + \psi'\left(k+\frac{1}{\beta}\right) + 2\beta\psi\left(k+\frac{1}{\beta}\right) + \\ \lambda^\beta x^\beta (\log x)^2 + \lambda^\beta (\log\lambda)^2 x^\beta + 2\lambda^\beta \log\lambda x^\beta \log x \end{array} \right) \tag{3.18}$$

Where $\Psi\left(k+\frac{1}{\beta}\right)$ is a di-gamma is function and $\Psi'\left(k+\frac{1}{\beta}\right)$ is a tri-gamma function.

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\lambda\partial\beta} = \frac{k-\lambda^\beta\left[(\beta\log\lambda+1)x^\beta - \beta\lambda^\beta x^\beta \log x\right]}{\lambda} \tag{3.19}$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k\partial\beta} = \log\lambda + \frac{\psi'\left(k+\frac{1}{\beta}\right) - \psi^2\left(k+\frac{1}{\beta}\right)}{\beta^2} \tag{3.20}$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k^2} = -\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} + \psi^2\left(k+\frac{1}{\beta}\right) \tag{3.21}$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\lambda\partial k} = \frac{\beta}{\lambda} \tag{3.22}$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\beta\partial k} = -\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} + \psi^2\left(k+\frac{1}{\beta}\right) \tag{3.23}$$

Where $\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} = \Psi''\left(k+\frac{1}{\beta}\right)$ is a tri-gamma function.

Taking expectations on both sides of the above equations, we get

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\lambda^2}\right) = \frac{(k\beta+1) + \beta(\beta-1)(\lambda)^\beta E(x)^\beta - 1}{\lambda^2} \qquad I(1,1)$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\beta\partial\lambda}\right) = \frac{-k+\lambda^\beta\left[(\beta\log\lambda+1)x^\beta + \beta E(x^\beta \log x)\right]}{\lambda} \qquad I(1,2)$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k\partial\lambda}\right)=-\frac{\beta}{\lambda} \qquad \text{I (1, 3)}$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\lambda\partial\beta}\right)=\frac{-k+\lambda^{\beta}\left[(\beta\log\lambda+1)E(x)^{\beta}-\beta\lambda^{\beta}E\left(x^{\beta}\log x\right)\right]}{\lambda} \qquad \text{I (2, 1)}$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\beta^2}\right)=\left(\begin{array}{l}\frac{1}{\beta^2}+\psi'\left(k+\frac{1}{\beta}\right)+2\beta\psi\left(k+\frac{1}{\beta}\right)+\lambda^{\beta}E\left[x^{\beta}(\log x)^2\right]+\\ \lambda^{\beta}(\log\lambda)^2E(x)^{\beta}+2\lambda^{\beta}\log\lambda E\left(x^{\beta}\log x\right)\end{array}\right) \qquad \text{I (2, 2)}$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k\partial\beta}\right)=\log\lambda+\frac{\psi'\left(k+\frac{1}{\beta}\right)-\psi^2\left(k+\frac{1}{\beta}\right)}{\beta^2} \qquad \text{I (2, 3)}$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\lambda\partial k}\right)=\frac{\beta}{\lambda} \qquad \text{I (3, 1)}$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial\beta\partial k}\right)=-\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}+\psi^2\left(k+\frac{1}{\beta}\right) \qquad \text{I (3, 2)}$$

$$-E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k^2}\right)=-\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}+\psi^2\left(k+\frac{1}{\beta}\right) \qquad \text{I (3, 3)}$$

We know that, $E\left(x^{\beta}\log x\right)=\int_0^{\infty} x^{\beta}\log x f_s(x;\lambda,\beta,k)dx$

$$E\left(x^{\beta}\log x\right)=\int_0^{\infty} x^{\beta}\log x\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\,e^{-(\lambda x)^{\beta}}\,dx$$

$$E\left(x^{\beta}\log x\right)=\frac{\Gamma'\left(k+\frac{1}{\beta}+1\right)-\beta\log\lambda\Gamma\left(k+\frac{1}{\beta}+1\right)}{\beta\lambda^{\beta}\Gamma\left(k+\frac{1}{\beta}\right)} \qquad (3.24)$$

Also, $E\left(x^{\beta}(\log x)^{2}\right)=\int_{0}^{\infty} x^{\beta}(\log x)^{2} f_{s}\ (x;\lambda,\beta,k)dx$

$$E\left(x^{\beta}(\log x)^{2}\right)=\int_{0}^{\infty} x^{\beta}(\log x)^{2}\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}\ dx$$

$$E\left(x^{\beta}(\log x)^{2}\right)=\frac{\Gamma''\left(k+\frac{1}{\beta}+1\right)-2\beta\log\lambda\Gamma'\left(k+\frac{1}{\beta}+1\right)+\beta^{2}(\log\lambda)^{2}\Gamma\left(k+\frac{1}{\beta}+1\right)}{\beta^{2}\lambda^{\beta}\Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.25}$$

Also, $$E(X^{\beta})=\frac{\Gamma\left(k+\frac{\beta+1}{\beta}\right)}{\lambda^{\beta}\Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.26}$$

Substituting the values of equations (3.24), (3.25) and (3.26) in the above entries of a Fisher information matrix, we get

$$I(1,1)=\frac{(k\beta+1)\Gamma\left(k+\frac{1}{\beta}\right)+\beta(\beta-1)\Gamma\left(k+\frac{1}{\beta}+1\right)-1}{\lambda^{2}\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(1,2)=\frac{-k\Gamma\left(k+\frac{1}{\beta}\right)+\left[\left(\beta\log\lambda-\beta^{2}\log\lambda+1\right)\Gamma\left(k+\frac{1}{\beta}+1\right)+\beta\Gamma'\left(k+\frac{1}{\beta}+1\right)\right]}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(1,3)=-\frac{\beta}{\lambda}$$

$$I(2,1)=\frac{-k\Gamma\left(k+\frac{1}{\beta}\right)+\left[\Gamma\left(k+\frac{1}{\beta}+1\right)+\left(\Gamma'\left(k+\frac{1}{\beta}+1\right)\right)\right]}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(2,2)=\frac{\left(\Gamma\left(k+\frac{1}{\beta}\right)+\beta^{2}\Gamma''\left(k+\frac{1}{\beta}\right)+2\beta^{3}\Gamma'\left(k+\frac{1}{\beta}\right)+\Gamma''\left(k+\frac{1}{\beta}+1\right)\right)}{\beta^{2}\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(3,1)=-\frac{\beta}{\lambda}$$

$$I(3,2)=-\log\lambda-\frac{1}{\beta^{2}}\left[\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}+\psi^{2}\left(k+\frac{1}{\beta}\right)\right]$$

$$I(3,3)=\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}-\psi^{2}\left(k+\frac{1}{\beta}\right)$$

3.5 Entropy estimation of Size-biased Generalized Gamma Distribution

Consider the Probability Density function of size biased generalized gamma distribution (1.1)

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\,e^{-(\lambda x)^{\beta}} \qquad (3.27)$$

$$\log L^{*}(X;\lambda,\beta,k)\;=\;n(1+k\beta)\log\lambda+n\log\beta-n\log\Gamma\left(k+\frac{1}{\beta}\right)+$$

$$k\beta\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta} \qquad (3.28)$$

$$l(X;\lambda,\beta,k)\;=n\left((1+k\beta)\log\lambda+\log\beta-\log\Gamma\left(k+\frac{1}{\beta}\right)\right)+k\beta\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta} \qquad (3.29)$$

$$\frac{l(x;\lambda,\beta,k)}{n}\;=\;(1+k\beta)\log\lambda+\log\beta-\log\Gamma\left(k+\frac{1}{\beta}\right)+k\beta\overline{\log x}-\lambda^{\beta}\overline{x^{\beta}} \qquad (3.30)$$

The Shannon's entropy Estimation of Size-biased Generalized Gamma Distribution is given as:

$$\hat{H}(SBGG)=-\left[(1+\hat{k}\hat{\beta})\log\hat{\lambda}+\log\hat{\beta}-\log\Gamma\left(\hat{k}+\frac{1}{\hat{\beta}}\right)+\hat{k}\hat{\beta}\overline{\log x}-\hat{\lambda}^{\hat{\beta}}\overline{x^{\hat{\beta}}}\right] \qquad (3.31)$$

Comparing equations (3.3) and (3.31), we can state that both equations are same, because $E(\log(x)) = \overline{\log(x)}$ and $E(x^{\beta})) = \overline{x^{\beta}}$. From equation (3.30) and (3.31), we can write

$$\hat{H}(SBGG) = -\frac{l(x;\hat{\lambda},\hat{\beta},\hat{k})}{n} \tag{3.32}$$

Where $\hat{H}(SBGG)$ is the Shannon's entropy estimation, $l(x;\hat{\lambda},\hat{\beta},\hat{k})$ is the logarithm likelihood, $\hat{\lambda}$ is the estimated value of λ, $\hat{\beta}$ is the estimated value of β and $\hat{k}$ is the estimated value of k of the size biased is generalized gamma distribution and n is the sample size. The estimators' like $\hat{\lambda},\hat{\beta}$ and $\hat{k}$ can be obtained by employing the maximum likelihood estimation.

3.6 Akaike and Bayesian information criterion

In order to introduce an approach for model selection, we remember Akaike and Bayesian information criterion based on entropy estimation. Akaike's information criterion, developed by Hirotsugu Akaike (1973) under the name of "an information criterion" (AIC) in 1971 and proposed in Akaike (1974), is a measure of the goodness of fit of an estimated statistical model. It is grounded in the concept of entropy, in effect offering a relative measure of the information lost when a given model is used to describe reality and can be said to describe the trade-off between bias and variance in model construction, or loosely speaking that of precision and complexity of the model. The AIC is not a test of the model in the sense of hypothesis testing; rather it is a test between models - a tool for model selection. Given a data set, several competing models may be ranked according to their AIC, with the one having the lowest AIC being the best. From the AIC value one may infer that e.g. the top three models are in a tie and the rest are far worse, but it would be arbitrary to assign a value above which a given model is "rejected". In the general case, the AIC is

$$AIC = 2K - 2\log L(\hat{\theta})$$

Where K is the number of parameters in the statistical model and L is the maximized value of the likelihood function for the estimated model.

The Bayesian information criterion (BIC) or Schwarz Criterion is a criterion for model selection among a class of parametric models with different numbers of parameters. Choosing a model to optimize BIC is a form of regularization. It is very closely related to AIC. In BIC, the penalty for additional parameters is stronger than that of the AIC.

The formula for the BIC is

$$BIC = K\log n - 2\log L(\hat{\theta})$$

The AIC and BIC methodology attempts to find the model that best explains the data with a minimum of their values, from (3.38) we have

$$l(x;\lambda,\beta,k) = -n\hat{H}(\text{SBGG})$$

Then for SBGG family we have

$$AIC = 2K + 2n\hat{H}(\text{SBGG}) \tag{3.33}$$

and $BIC = K\log n + 2n\hat{H}(\text{SBGG})$ (3.34)

3.7 Test for Size-biasedness of Size biased Generalized Gamma Distribution

Let $x_1, x_2 \ldots x_n$ be random samples can be drawn from Generalized Gamma Distribution or Size biased Generalized Gamma distribution. We test the hypothesis

$H_0 : f(x) = f(x;\lambda,\beta,k)$ againest $H_1 : f(x) = f_S(x;\lambda,\beta,k)$

To test whether the random sample of size n comes from the generalized Gamma distribution or Size biased generalized Gamma distribution, then the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{f_S(x;\lambda,\beta,k))}{f(x;\lambda,\beta,k))}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}\, e^{-(\lambda x)^{\beta}}}{\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\, e^{-(\lambda x)^{\beta}}}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{\lambda\Gamma(k)}{\Gamma\left(k+\frac{1}{\beta}\right)}x\right].$$

$$\Delta = \left[\frac{\lambda\Gamma(k)}{\Gamma\left(k+\frac{1}{\beta}\right)}\right]^n \prod_{i=1}^{n} x_i \tag{3.35}$$

We reject the null hypothesis.

$$\left[\frac{\lambda\Gamma(k)}{\Gamma\left(k+\frac{1}{\beta}\right)}\right]^n \prod_{i=1}^{n} x_i > k \tag{3.36}$$

For some constant k. Equivalently, we reject the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where\, k^* = k\left[\frac{\lambda\Gamma(k)}{\Gamma\left(k+\frac{1}{\beta}\right)}\right]^n > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution. Also, we can reject the reject the null hypothesis, when probability value s given by:

$P(\Delta^* > \lambda^*)$, Where $\lambda^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the test statistic.

4. Estimation of parameters in the size-biased Generalized Gamma Distribution.

In this section, we obtain estimates of the parameters for the Size-biased Generalized Gamma distribution by employing the method of moment (MOM) and maximum likelihood (ML) estimators.

4.1 Method of Moment Estimators of size-biased Generalized Gamma Distribution

Let $X_1, X_2, X_3 \ldots X_n$ be an independent random samples from the Size-biased Generalized Gamma (SBGG) distribution with weight c=1. The method of moment estimators are obtained by setting the raw moments equal to the sample moments, that is $E(X_r) = M_r$ where is the sample moment Mr corresponding to the $E(X_r)$.The following equations are obtained using the first and second sample moments.

$$\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)} = \frac{1}{n}\sum_{i=1}^{n} X_j \tag{4.1}$$

$$\frac{\Gamma\left(k+\frac{3}{\beta}\right)}{\lambda^2\Gamma\left(k+\frac{1}{\beta}\right)} = \frac{1}{n}\sum_{i=1}^{n} X_j^2 \tag{4. 2}$$

4.1.1 When β and k are fixed and from equation (4.1), we obtain an estimate $\hat{\lambda}$ for λ, that is

$$\hat{\lambda} = \frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\bar{X}\,\Gamma\left(k+\frac{1}{\beta}\right)} \tag{4.3}$$

4.1.2 When $\beta = 1$ and λ are fixed and dividing (4.1) by equations (4.2), we get

$$\hat{k} = \frac{M_2}{\bar{X}}.\lambda - 2 \tag{4.4}$$

4.1.3 When λ and k are fixed, the estimate for β can be obtained by numerical methods.

4.2 Maximum likelihood Estimator of size-biased Generalized Gamma Distribution

Let $x_1, x_2, ..., x_n$ be a random sample from a Size-biased Generalized Gamma Distribution. Then the likelihood function of Size-biased Generalized Gamma (SBGG) Distribution is given by:

$$L(X;\lambda,\beta,k) = \prod_{i=1}^{n} f_s(x;\lambda,\beta,k)$$

$$L(X;\lambda,\beta,k) = \frac{\lambda^{n(1+k\beta)}\beta^n}{\Gamma^n\left(k+\frac{1}{\beta}\right)} \prod_{i=1}^{n} x_i^{k\beta} e^{-\lambda^\beta \sum_{i=1}^{n} x_i{}^\beta} \tag{4.5}$$

Using equation (4.5), the log likelihood function is given by

$$\log L^*(X;\lambda,\beta,k) = n(1+k\beta)\log\lambda + n\log\beta - n\log\Gamma\left(k+\frac{1}{\beta}\right) + k\beta\sum_{i=1}^{n}\log x_i - \lambda^\beta \sum_{i=1}^{n} x_i^\beta \tag{4.6}$$

Now differentiate $\log L^*(X;\lambda,\beta,k)$ with respect to λ, β and k, we get

$$\frac{\partial L^*}{\partial\lambda} = \frac{n(1+k\beta)}{\lambda} - \sum_{i=1}^{n} x_i{}^\beta \beta\lambda^{\beta-1}$$

$$\frac{\partial L^*}{\partial\beta} = nk\log\lambda + \frac{n}{\beta} + \frac{n\Psi\left(k+\frac{1}{\beta}\right)}{\beta^2} + k\sum_{i=1}^{n}\log x_i - \left(\lambda\sum_{i=1}^{n} x_i\right)^\beta \left(\log\sum_{i=1}^{n} x_i + \log\lambda\right)$$

$$\frac{\partial L^*}{\partial k} = n\beta\log\lambda - n\Psi\left(k+\frac{1}{\beta}\right) + \beta\sum_{i=1}^{n}\log x_i$$

Equating these equations to zero, leads to the normal equations:

$$\frac{n(1+k\beta)}{\lambda}-\sum_{i=1}^{n} x_i^{\beta}\beta\lambda^{\beta-1}=0 \qquad (4.7)$$

$$nk\log\lambda+\frac{n}{\beta}+\frac{n\Psi\left(k+\frac{1}{\beta}\right)}{\beta^2}+k\sum_{i=1}^{n}\log x_i-\left(\lambda\sum_{i=1}^{n}x_i\right)^{\beta}\left(\log\sum_{i=1}^{n}x_i+\log\lambda\right)=0 \qquad (4.8)$$

$$n\beta\log\lambda-n\Psi\left(k+\frac{1}{\beta}\right)+\beta\sum_{i=1}^{n}\log x_i=0 \qquad (4.9)$$

4.2.1 When β and λ are fixed, It follows from equation (4.7), that

$$\hat{k}=\frac{\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta}}{n}-\frac{1}{\beta} \qquad (4.10)$$

4.2.2 When β and k are fixed, It follows from equation (4.9), that

$$\hat{\lambda}=\left[\exp\left(\Psi\left(k+\frac{1}{\beta}\right)-\frac{\beta\sum_{i=1}^{n}\log x_i}{n}\right)\right]^{\frac{1}{\beta}} \qquad (4.11)$$

4.2.3 When λ and k are fixed, the estimate for β can be obtained by numerical methods.

5. Concluding Remark

The objective of this article is to obtain and derive the various structural properties, reliability measures and information measures of size biased generalized gamma distribution. Also a likelihood ratio test for size-biasedness is conducted. The estimation of parameters is obtained by employing the classical estimators especially methods of moments and maximum likelihood estimators. The future research may be considered to develop a mixture of Generalized Gamma (GG) distribution with the Size biased Generalized Gamma (SBGG) distribution. Also, it may be consider to estimate the parameters of the new model by using different loss functions especially LINLEX loss function, Quadratic, precautionary loss function and generalized entropy Loss function under different prior distributions like Gamma prior distributions, Conjugate priors and double priors etc.

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References

1. Ahmed, A., Mir, K.A. and Reshi, J. A. (2013a). On new method of estimation of parameters of Size-biased Generalized Gamma Distribution and its Structural properties, IOSR Journal of Mathematics, Volume 5(2), p. 34-40.
2. Ahmed, A, Reshi, J. A and Mir, K. A. (2013b). Structural properties of Size-biased Gamma Distribution, IOSR Journal of Mathematics, 5(2), p. 55-61.
3. Akaike, H. (1973). Information Theory as an extension of the maximum likelihood principle, Second International Symposium on Information Theory, Akademiai Kiado, Budapest.
4. Akaike, H. (1974). A new look at the statistical model identification, IEEE Transactions on Automatic Control, 19(6), p. 7-16.
5. Golan, Amos. (2006). A Review and synthesis, foundations and trends in Econometrics, Information and Entropy measures, 2, p. 1-2.
6. Huang, P. H. and Hwang, T. Y. (2006). On new moment estimation of parameters of the generalized gamma distribution using it's characterization, Taiwanese Journal of Mathematics, Volume 10(4), p. 1083-1093.
7. James Shannon C.E. (1948). A Mathematical theory of communication, Bell system Technical Journal, 27, p. 623-659.
8. Kapur, J. N. (1989). Maximum Entropy Models in Science and Engineering, Wiley, New York.
9. Mir, K.A, Ahmed, A. and Reshi, J. A. (2013). On Size-biased Exponential Distribution, Journal of Modern Mathematics and Statistics,7 (2), p. 21-25.
10. Prentice, R. L. (1974). A Log gamma model and its maximum likelihood estimation, Biometrika, 61, p.539-544.
11. Reshi, J. A., Ahmed, A. and Mir, K. A. (2014). On new moment method of estimation of parameters of size-biased classical gamma distribution and its characterization, International Journal of Modern Mathematical Sciences, 10(2), p. 179-190.
12. Reshi, J. A., Ahmed, A. and Mir, K. A. (2014). Bayesain estimation of size biased classical gamma distribution, Journal of Reliability and Statistical Studies, 7(1), p. 31-42.