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SOMETIMES POOL TESTIMATION OF SCALE PARAMETER FOR NEGATIVE EXPONENTIAL MODEL UNDER GENERAL ENTROPY LOSS FUNCTION

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Abstract

The present paper proposes a sometimes pool testimator of scale parameter (mean life) of negative exponential distribution under general entropy loss function, when it is assume that both the guarantees are known If a real life situation is modeled by this distribution having mean life time of certain items as θ_1 and now it is suspected that the mean life may change due to some technological advances and assumes the value θ_2 then we may have conditional information on θ_1 as $\theta_1 \geq \theta_2$(however θ_1 may be less than or equal to θ_2).This uncertainty can be resolved by using preliminary testing and then a sometimes pool testimator is proposed for θ_1The risk properties of this estimator have been studied under General entropy loss function and it is claimed that the estimator dominates the never pool estimator (in terms of having smaller risk) in certain range of life ratio. Use of a general entropy loss function facilitates to control the risk of proposed estimators for various directions and degrees of asymmetry.

Key Words and Phrases: Negative exponential distribution, Scale Parameter, Type – II Censored Samples, Preliminary Test, Level of Significance, General Entropy Loss Function, Life Ratio, Euler's Psi Function, Relative Risk.

1. Introduction

1.1 The model

The negative exponential distribution has been a subject of comprehensive studies since fifties. A systematic development of life testing originated from the work of Epstein and Sobel (1953) and the subsequent progress made in this field can gauged from the bibliography of Mendenhall (1958) and Govindrajulu (1964) among others. Let us consider that two independent random sample sizes n_1 and n_2 are available from the negative exponential distributions defined by,

$$f(X_{i,}A_i,\theta_i) = \begin{cases} (\theta_i)^{-1} exp\left\{\frac{X_i - A_i}{\theta_i}\right\}; \; X_i & \geq A_i, \; \theta_1 > 0 \\ 0 & o.w. \end{cases} \tag{1.1.1}$$

Where A_i and θ_i $(i = 1, 2)$ are the location and scale parameter respectively. A_i is also interpreted as the minimum guarantee period or threshold or shift parameter or within which no failure can take place, and θ_i represents the mean life.

Under the assumption of squared error loss function (SELF) Ramkaran and Bhattacharya (1984) have estimated the mean life of negative exponential distribution by a sometimes pool estimator. Rai (1996) proposed a sometimes pool estimator for mean life under Linex loss function, for a single parameter exponential distribution. Srivastava and Tank (2001) have studied the risk properties of Sometimes pool testimators using an asymmetric loss function proposed by Basu and Ebrahimi.
In this paper we have considered the Negative Exponential distribution and proposed an estimator of θ_1, using type II censored sample information. The problem under consideration has the following three situations.

Situation I: A_1 and A_2 are known.

Situation II: $A_1 = A_2$ and the common value may be known /unknown.

Situation II: Nothing is known about A_1 and A_2.

The present paper deals with situation I only.

1.2 Asymmetric loss functions

The choice of an appropriate loss function is very important under the Bayesian set up as it is the loss function chosen, decides about the Bayes estimator and more importantly the risk associated with it. Several authors have proposed Bayes estimators for different parameter(s) in various distributions using different loss functions.

We know that in many real life situations the overestimation or under estimation are not of equal consequences. Several authors such as Canfield (1975), Zellner (1986), Basu and Ebrahimi (1991), Srivastava (1996), Srivastava and Tank (2001), Srivastava and Tanna (2001), Srivastava and Tanna (2007, 2012), Srivastava and Shah (2010) and others have shown that the estimators or testimators of the parameters of interest under the asymmetric loss function demonstrate their superiority over the estimators obtained under squared error loss function (SELF).

Basu and Ebrahimi (1991) proposed a modified LINEX loss function, though this worked for many parameter(s) and parametric functions but had its own limitations in some cases where the estimators could not be obtained in closed forms.
A suitable alternative to modified LINEX loss is the General Entropy Loss (GEL) proposed by Calabria and Pulcini (1994 a) given by:

$$L_E(\widehat{\theta}, \theta) \propto \left\{ \left(\frac{\widehat{\theta}}{\theta}\right)^p - p\ln\left(\frac{\widehat{\theta}}{\theta}\right) - 1 \right\} \qquad (1.2.2)$$

Whose minimum occurs at$\hat{\theta} = \theta$.

This loss is a generalization of the entropy loss used by several authors for example, Dey et.al (1987), and Dey and Liu (1992), where the shape parameter ‘p’ is equal to unity (1). The more general version of (1.2.2) allows different shapes of the loss function to be considered (say) when $p > 0$, a positive error ($\hat{\theta} > \theta$) causes more serious consequences than a negative error and similarly when p<0, a negative error is more serious.

The Bayes estimate of θ under the GEL can be obtained in a closed form using:

$$\hat{\theta}^{E} = [E_{\theta}(\theta^{-p})]^{\frac{1}{p}} \qquad (1.2.3)$$

Provided that $E_{\theta}(\theta^{-p})$ exists and is finite.

When $p = -1$, the bayes estimate (1.2.3) coincides with the Bayes estimate under the squared error loss function.

1.3 Background

Srivastava and Tank (2001) have considered the problem of testimation of a scale parameter with probability density function given by (1.1.1) under linex loss function, when it is assumed that both the guarantees are known. We have proposed a sometimes pool estimator of scale parameter (θ_1) in section 2. The risk of the proposed estimator $\hat{\theta}_{SP}$ using GEL function has been derived in section 3. In section 4 we have compared the relative risk of $\hat{\theta}_{SP}$ with never pool estimator$\hat{\theta}_{N}$. The paper concludes with section 5.

2. The sometimes pool testimator

Let $X_{11} \leq X_{12} \leq \cdots \leq X_{1r_1}$ be the ordered failure times of the first r_1 items in a life testing experiment, in which n_1 items were placed on test and let $X_{21} \leq X_{22} \leq \cdots \leq X_{2r_2}$ be the ordered failure times of first r_2 items in another life test, where n_2 items were placed. Suppose that the underlying distribution of each X_{ij} is a two parameter exponential distribution $f(X_i, A_i, \theta_i),\ i = 1,2;\ j = 1,2, \ldots, r_i$ given by (1.1.1).

We are interested in estimatingθ_1, when it is suspected that there may be a change in the average life($i.e.\ \theta_1 \geq \theta_2$). Then to incorporate this doubtful information, we test the hypothesis $H_0\colon \theta_1 = \theta_2$ against $H_1\colon \theta_1 \neq \theta_2$ using the test statistic proposed by Epstein and Sobel (1953).As the two samples are combined only when H_0 is accepted implying that a pooled estimator can be proposed thus we have the always pool estimator. However when H_0 is rejected we do not have sufficient evidence to combine the two samples and hence work with only the first sample so we have a never pool estimator. Thus, our Sometimes pool estimator can be proposed as follows:

$$\hat{\theta}_{SP} = \begin{cases} \hat{\theta}_p = \frac{u_1+u_2}{r_1+r_2}; & if\ H_0\ is\ accepted \\ \hat{\theta}_N = \frac{u_1}{r_1}; & 0.w. \end{cases} \quad (2.1)$$

Where, $u_i = \sum_{j=1}^{r_i}(X_{ij} - A_i) + (n_i - r_i)(X_{ir_i} - A_i);\ i = 1, 2.$

We know that $\frac{2u_1}{\theta_1}$ and $\frac{2u_2}{\theta_2}$ follow χ^2 distribution with $2r_1$ and $2r_2$ degrees of freedom respectively.

To test H_0, we use the statistic $F = \frac{r_2}{r_1}.\frac{u_1}{u_2}$ which follows F distribution with $(2r_1, 2r_2)$ degrees of freedom. So, the required test is to reject H_0 at $100\alpha\%$ level of significance, if $F < F_1$ or $F > F_2$ where $F_1 = F_1(2r_1, 2r_2)$ and $F_2 = F_2(2r_1, 2r_2)$ are the lower and upper $\frac{\alpha}{2}$ quantiles of F distribution with $2r_1$ and $2r_2$ degrees of freedom.

Thus $\hat{\theta}_{SP}$ can be written as:

$$\hat{\theta}_{SP} = \begin{cases} \hat{\theta}_p = \frac{u_1+u_2}{r_1+r_2}; & if\ F_1 \le \frac{r_2}{r_1}.\frac{u_1}{u_2} \le F_2 \\ \hat{\theta}_N = \frac{u_1}{r_1}; & 0.w. \end{cases} \quad (2.2)$$

The risk of $\hat{\theta}_{SP}$ has been derived in the next section.

3. Risk of $\hat{\theta}_{SP}$

The joint density of u_1 and u_2 is given by

$$g(u_1, u_2) = c_1(u_1)^{r_1-1}(u_2)^{r_2-1}exp\left\{-\left(\frac{u_1}{\theta_1} + \frac{u_2}{\theta_2}\right)\right\}$$

Where$c_1 = (\Gamma(r_1)\Gamma(r_2)(\theta_1)^{r_1-1}(\theta_2)^{r_2-1})^{-1}$

Now let us make following transformations

$$x = u_1 + u_2 \rightarrow u_1 = \frac{xy}{1+y}$$
$$y = \frac{u_1}{u_2} \rightarrow u_2 = \frac{x}{1+y} \text{ with jacobian } |J| = \frac{x}{(1+y)^2} \quad (3.1)$$

Then the joint density functions of $x$and y is;

$$g(x, y) = c_1\frac{x^{r_1+r_2-1}y^{r_1-1}}{(1+y)^{r_1+r_2}}exp\left\{-\frac{x}{1+y}\left(\frac{y}{\theta_1} + \frac{1}{\theta_2}\right)\right\} \quad (3.2)$$

Again let us make the following transformations

$$u_1 = u_1, y = \frac{u_1}{u_2} \rightarrow u_2 = \frac{u_1}{y} \text{ with jacobian } |J| = \frac{u_1}{y^2} \tag{3.3}$$

Therefore, the joint density function of (u_1, y) is;

$$g(u_1, y) = c_1 \frac{{u_1}^{r_1+r_2-1}}{y^{r_2+1}} exp\left\{-\frac{u_1}{y}\left(\frac{y}{\theta_1}+\frac{1}{\theta_2}\right)\right\} \tag{3.4}$$

Now, the risk of $\hat{\theta}_{SP}$ under general entropy loss function $L_E(\hat{\theta}, \theta)$ can be defined as:

$$R_E(\hat{\theta}_{SP}) = \left[\begin{array}{c} \left\{\int_{F_1}^{F_2}\int_0^{\infty}\left[\left\{\frac{\hat{\theta}_P}{\theta_1}\right\}^p - pln\left\{\frac{\hat{\theta}_P}{\theta_1}\right\} - 1\right] g(x,y)dxdy\right\} \\ + \left\{\int_0^{\infty}\int_0^{\infty}\left[\left\{\frac{\hat{\theta}_N}{\theta_1}\right\}^p - pln\left\{\frac{\hat{\theta}_N}{\theta_1}\right\} - 1\right] g(u_1)g(y)du_1dy\right\} \\ - \left\{\int_{F_1}^{F_2}\int_0^{\infty}\left[\left\{\frac{\hat{\theta}_N}{\theta_1}\right\}^p - pln\left\{\frac{\hat{\theta}_N}{\theta_1}\right\} - 1\right] g(u_1)g(y)du_1dy\right\} \end{array}\right] \tag{3.5}$$

A straightforward integration of (3.5) gives us

$$R_E(\hat{\theta}_{SP}) = \left[\begin{array}{c} \frac{\Gamma(r_1+r_2+p)\varphi^p}{\Gamma(r_1)\Gamma(r_2)(r_1+r_2)^p}\int_{\frac{r_1}{r_2}F_1}^{\frac{r_1}{r_2}F_2}\frac{y^{r_1-1}(1+y)^p}{(1+\varphi y)^{r_1+r_2+p}}dy \\ +p\left(\left(ln(r_1+r_2) - ln(r_1)\right)\left(I_{X_2}(r_1, r_2) - I_{X_1}(r_1, r_2)\right)\right) \\ +\frac{\Gamma(r_1+p)}{\Gamma(r_1)(r_1)^p}p\left(\Psi(r_1+r_2) - ln(r_1)\right) \\ +p\left(\Psi(r_1+r_2) - \Psi(r_1)\right) - 1 \\ -\frac{\Gamma(r_1+p)}{\Gamma(r_1)(r_1)^p}I_{X_2}\left(r_{1+p}, r_2\right) \\ -p\frac{\varphi^{r_1}}{\beta(r_1,r_2)}\int_{\frac{r_1}{r_2}F_1}^{\frac{r_1}{r_2}F_2} ln\left(\frac{1+\varphi y}{\varphi y}\right)\frac{y^{r_1-1}}{(1+\varphi y)^{r_1+r_2}}dy \end{array}\right] \tag{3.6}$$

Where,

$$x_1 = \frac{r_1 F_1 \varphi}{r_2 + r_1 F_1 \varphi}, \qquad x_2 = \frac{r_1 F_2 \varphi}{r_2 + r_1 F_2 \varphi}$$

$\Psi(n)$ is Euler's Psi function.

$\Psi(n) = \frac{d}{dx} ln(\Gamma(n))$, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

and $I(u,p) = \frac{\int_0^u e^{-x}x^{p-1}dx}{\int_0^{\infty} e^{-x}x^{p-1}dx}$; refers to the standardized incomplete gamma function.

4. Risk Comparison

A natural way of comparing the risk of the proposed sometimes pool testimator $\hat{\theta}_{SP}$, is to study its performance with respect to the best available estimator $\hat{\theta}_N$ for this purpose we obtain the risk of conventional estimator $\hat{\theta}_N$ under $L_E(\hat{\theta},\theta)$ as:

$$R_E(\hat{\theta}_N) = \left[\left\{\int_0^\infty \int_0^\infty \left[\left\{\frac{\hat{\theta}_N}{\theta_1}\right\}^p - pln\left\{\frac{\hat{\theta}_N}{\theta_1}\right\} - 1\right] g(u_1)g(y)du_1dy\right\}\right] \quad (4.1)$$

A straightforward integration of (4.1) gives us:

$$R_E(\hat{\theta}_N) = \begin{bmatrix} \frac{\Gamma(r_1+p)}{\Gamma(r_1)(r_1)^p} - p\left(\Psi(r_1+r_2) - ln(r_1)\right) \\ +p\left(\Psi(r_1+r_2) - \Psi(r_1)\right) - 1 \end{bmatrix} \quad (4.2)$$

Now define the relative risk of $\hat{\theta}_{SP}$ with respect to $\hat{\theta}_N$ under $L_E(\hat{\theta},\theta)$ as follows

$$R_R = \frac{R_E(\hat{\theta}_N)}{R_E(\hat{\theta}_{SP})} \quad (4.3)$$

Using (3.6) and (4.2) the expression for R_R given in (4.3) can be obtained; it is observed that R_R is a function of r_1 ,r_2, p, φ and $\propto$. To observe the behavior of R_R we have taken several values of these viz. $\propto = 1\%, 5\%, 10\%$ and 16%, φ represents the life ratio and has been allowed to vary as $\varphi = 0.2(0.2)1.0$. 'p' which is the shape parameter for the loss function is prime important factor and decides about the overestimation/under estimation in the real life situation has been taken as $p = \pm 1.0(\pm 1.5) \pm 3.0$. We have considered several values of r_1 and r_2 which are given in the following tables (table 4.1 when $r_1 < r_2$ and table 4.2 when$r_1 > r_2$) to observe the performance ofR_R. Some graphs of R_R for the data considered above are provided in the appendix. However, our conclusions based on all the graphs are given in the next section.

Table 4.1 ($r_1 < r_2$)		**Table 4.2 ($r_1 > r_2$)**	
r_1	r_2	r_1	r_2
4	8	8	4
6	8	8	6
8	10	10	8

5. Conclusion

The relative risk calculations for these different values of r_1, r_2, p, φ and $\propto$ are made and these values show that the proposed testimator $\hat{\theta}_{SP}$ of scale parameter (θ_1) fairs better than the unbiased estimator of the same for almost all values of φ (life ratio) considered here. Practically whenever φ (life ratio) is greater than or equal to 0.5

it definitely performs better. While using the sometimes pool estimator a quantity of interest is the level of significance 'α' and while using the asymmetric loss functions the quantity of interest is the degree of asymmetry, so here we have fixed 'α' and allowed the variations in ps and then we have fixed 'p' and allowed the variations in $\propto$s. Values of R_R is maximum at $\propto = 1\%$ and when $p = 3.0$, however, for other values of 'p' also the values of $R_R > 1$ indicating that the proposed estimator is better. Next we consider the negative values of 'p' and it is observed that for $p = -1.0$ &-3.0 and $\propto = 1\%$ the maximum gain is obtained in the values of R_R, it is also noticed that for other negative values of 'p' the behaviour is same i.e. $\hat{\theta}_{SP}$ dominates $\hat{\theta}_N$ but the reported values correspond to the maximum gain (in terms of relative risk). It is also to be remarked that behaviour is same for $r_1 > r_2$ and $r_1 < r_2$ situations however it is noticed that the magnitude of R_R is lower in case for $r_1 > r_2$.

So, in the present study it is concluded that a lower level of significance i.e.$\propto = 1\%$ yields better results when overestimation is more harmful than underestimation i.e. $p = 3.0$, however the same level seems to be better for $p = -1.0$ &-3.0 (but now with a changed magnitude of degree of asymmetry). Therefore estimation of $\hat{\theta}_{SP}$ using general entropy loss function gives better control over error in both over estimation/underestimation situations.

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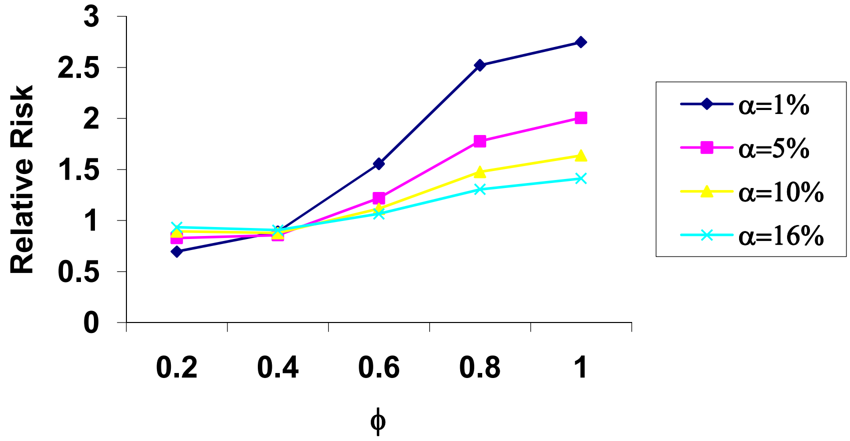
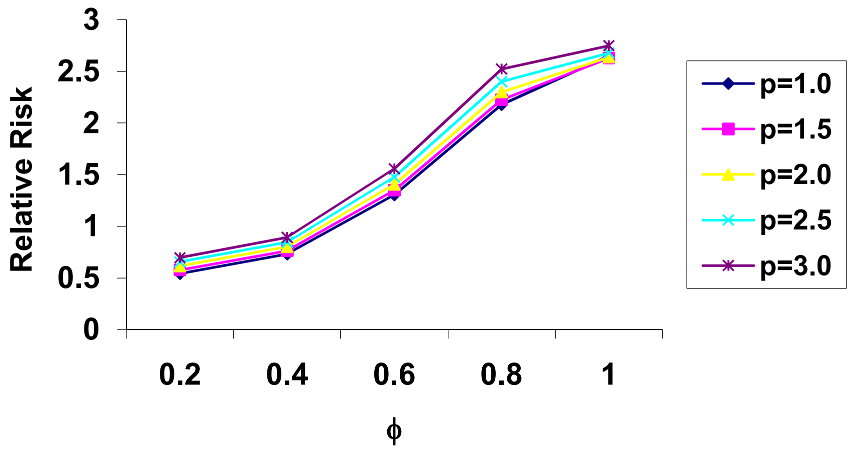
Appendix

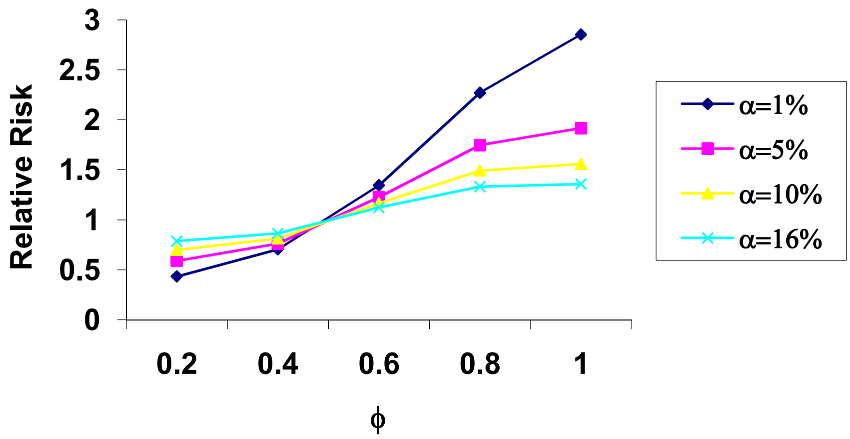
(1) $r_1 = 4, \quad r_2 = 8, \quad p = 3.0$



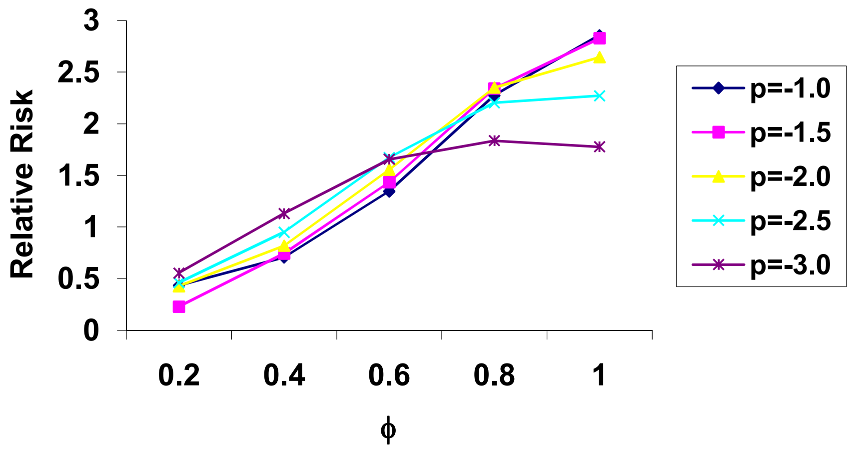
(2) $r_1 = 4, \quad r_2 = 8, \quad \alpha = 1\%$



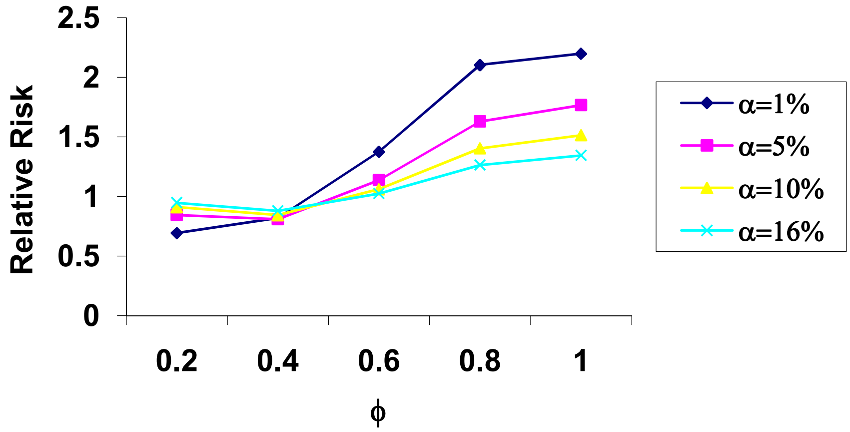
(3) $r_1 = 4, \quad r_2 = 8, \quad p = -1.0$



(4) $r_1 = 4, \quad r_2 = 8, \quad \alpha = 1\%$



(5) $r_1 = 6, \quad r_2 = 8, \quad p = 3.0$



(6) $r_1 = 6, \quad r_2 = 8, \quad \alpha = 1\%$