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BAYES ESTIMATION OF SCALE PARAMETER IN GENERALIZED PARETO DISTRIBUTION

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Abstract

Bayes estimators of the scale parameter (p) with known location parameter (μ) and fixed shape parameter (k) of generalized Pareto model are obtained for different priors using Squared Error Loss Function (SELF) and Asymmetric Precautionary Loss Function (APLF) through Lindley's approach. The calculations have been illustrated with the help of a real data set.

Key Words: Bayes Estimator, Prior, Scale Parameter, Squared Error Loss Function and Asymmetric Precautionary Loss Function.

1. Introduction

The Pareto distribution is not limited to describe wealth or income distribution, but to many other situations. It may be used approximately in describing the situations such as the frequencies of words in longer texts, the size of human settlements (few cities, many hamlets/villages), file size distribution of internet traffic which uses the TCP protocol (many smaller files, few larger ones), the value of oil reserves in oil fields (a few large fields, many small fields), the length distribution in jobs assigned supercomputers (a few large ones, many small ones), the standardized price returns on individual stocks, size of sand particles, size of meteorites, number of species per genus (the tendency to divide a genus into two or more increases with the number of species in it), areas burnt in forest fires etc.

There are different forms of Pareto distribution. We have considered generalized Pareto distribution. We have obtained the Bayes estimators of the scale parameter (p) with known location parameter (μ) and fixed shape parameter (k) of generalized Pareto model for different priors using Squared Error Loss Function (SELF) and Asymmetric Precautionary Loss Function (APLF) through Lindley's approach.

Probability density function of Generalized Pareto model is

$$f(x/\mu, p, k) = \frac{1}{p}\left[1 + \frac{k(x-\mu)}{p}\right]^{-\frac{1}{k}-1} \qquad k \neq 0$$
$$= \frac{1}{p}\exp\left[-\frac{(x-\mu)}{p}\right] \qquad k = 0 \tag{1}$$

Domain

$$\mu \le x < \infty \qquad for\, k \ge 0$$

$$\mu \le x \le \mu - \frac{p}{k} \qquad for\, k < 0$$

where k, p and μ are shape, scale and location parameters respectively.

According to Hosking and Wallis [6] the generalized Pareto distribution is a two-parameter distribution that contains uniform, exponential and Pareto distributions as special cases. It has applications in a number of fields, including reliability studies and the analysis of environmental extreme events. Singh and Guo [15] employed the principle of maximum entropy (POME) to derive a new method of parameter estimation for the 3-parameter generalized Pareto distribution. The parameter estimates yielded by the POME were either superior or comparable for high skewness.

Castillo and Hadi [2] proposed a method for estimating the parameters and quantiles of the generalized Pareto distribution which gives well defined and easily computable estimators for all parameter values. Cheng and Chou [3] derived the expected value, variances and covariance's of the order statistics from the generalized Pareto distribution and obtained the best linear unbiased estimate of the scale parameter based on a few order statistics selected from a complete sample or a type-II censored sample. Bermudez and Turkman [1] used several methods for estimating the parameters of the generalized Pareto distribution (GPD), namely maximum likelihood (ML), the method of moments (MOM) and the probability-weighted moments (PWM).

Oztekin [10] compared the parameter estimation methods of the moments, probability-weighted moments, maximum likelihood, principle of maximum entropy and least squares to estimate the parameters in the three-parameter generalized Pareto distribution. Deidda and Puliga [5] assumed that the generalized Pareto distribution (GPD) can reliably represent the distribution of daily rainfall depths. Pandey and Rao [11] obtained Bayes estimators of the shape parameter of the generalized Pareto distribution by taking quasi, inverted gamma and uniform prior distributions using the linex, precautionary and entropy loss functions. These were compared with the corresponding Bayes estimators under the squared error loss function.

Lee [7] focussed on modelling and estimating tail parameters of loss distributions from Taiwanese commercial fire loss severity. Using extreme value theory, he employed the generalized Pareto distribution (GPD) and compared it with standard parametric modelling based on lognormal, exponential, gamma and Weibull distributions. Danish and Aslam [4] dealt with the Bayesian estimation of generalized exponential distribution in the proportional hazards model of random censorship under asymmetric loss functions. They stated that it is well known for the two-parameter lifetime distributions that the continuous conjugate priors for parameters do not exist. They assumed independent gamma priors for the scale and the shape parameters. It was observed by them that the closed-form expressions for the Bayes estimators cannot be obtained, therefore they have used Tierney–Kadane's approximation and Gibbs sampling to approximate the Bayes estimates.

Mahmoud et al. [9] derived the maximum likelihood (ML) and the Bayes estimators for the two unknown parameters of the generalized Pareto (GP) distribution based on a new type of censoring scheme called a progressive first-failure censored. The approximate Bayes estimators were obtained under the assumptions of informative and non-informative priors. It was revealed that the Bayes estimators based on non informative and informative priors perform much better than the MLEs in terms of biases and MSEs. Setiya and Kumar [13] obtained Bayes estimators of the shape parameters of a Pareto type-I model for different priors using Squared Error and Asymmetric Precautionary Error Loss Functions through direct method and Lindley's approach.. Setiya et al. [14] obtained the Bayes estimators of the shape parameter of a Pareto type-II model for different priors using Squared Error Loss function and Asymmetric Precautionary Loss Functions through Lindley's approach.

Saxena et al. [12] stated that the Mukherjee Islam Failure Model is considered as a simple model to assess component reliability and may exhibit a better fit for failure data and also provide more appropriate information about hazard rate. They obtained the reliability computation and Bayesian estimation of system reliability when the applied stress and strength follows the Mukherjee Islam Failure Model.

1. Bayes Estimation

Suppose $X_1, X_2, \_\_\_\_\_\_\_, X_n$ is a random sample from a probability density function $f(x|p)$, where p is a value of the random variable Θ with known density $g(p)$. The p.d.f. $f(x|p)$ is regarded as a conditional p.d.f. of X given p where the marginal p.d.f. of p is given by $g(p)$. Thus the joint p.d.f. of ($(X_1, X_2, \_\_\_\_\_\_\_, X_n; p)$ is given by

$$H(x_1, x_2, \_\_\_\_\_\_\_, x_n; p) = \left\{\prod_{i=1}^{n} f(x_i / p)\right\} g(p)$$

$$= L(x_1, x_2, \_\_\_\_\_\_\_, x_n \mid p)\, g(p) \tag{2}$$

The marginal p.d.f. of ($X_1, X_2, \_\_\_\_\_\_\_, X_n$) is given by

$$P(x_1, x_2, \_\_\_\_\_\_\_, x_n) = \int_{\Theta} H(x_1, x_2, \_\_\_\_\_\_\_, x_n; p)\, dp$$

$$= \int_{\Theta} L(x_1, x_2, \_\_\_\_\_\_\_, x_n | p) g(p) dp \tag{3}$$

And the conditional p.d.f. of p given the data ($x_1, x_2, \_\_\_\_\_\_\_, x_n$) is given by

$$\prod H_0\left(p|x_1,x_2,\_\_\_\_\_\_,x_n\right) = \frac{H(x_{1,}x_2,\_\_\_\_,x_n;p)}{P(x_1,x_2,\_\_\_\_\ x_n)}$$

$$=\frac{L(x_1,x_2,\_\_\_\_,x_n|p)g(p)}{\int\limits_{\Theta} L(x_1,x_2,\_\_\_\_,x_n|p)g(p)dp} \tag{4}$$

This is known as posterior distribution of p. Once the posterior distribution has been obtained, it becomes the main object of study.

2. Lindley's approach

Bayes estimators are generally obtained as the ratio of two integrals which cannot be solved directly by using asymptotic expansion and calculus of difference. An approximate procedure for solving the ratio of such integrals was given by Lindley [8] as well as Tierney and Kadane [16]. We have used Lindley's approach for solving the ratio of integrals (I), where

$$I = \frac{\int\limits_{p} h(p)\, l(p / \underset{\wedge}{x}) g(p) dp}{\int\limits_{p} l(p / \underset{\wedge}{x}) g(p) dp}$$

The approximate solution of I is

$$I \approx h(p^*) + \frac{\sigma^{*2}}{2}\left[h_2(p^*) + 2h_1(p^*)u_1(p^*)\right] + \frac{\sigma^{*4}}{2}\left[L_3(p^*)h_1(p^*)\right] \tag{5}$$

where p^* is the MLE of p and h(p) is the function of p whose Bayes estimator is to be obtained.

here,

$$L_k(p^*) = \left.\frac{\partial^k}{\partial p^k} L(p)\right|_{p=p^*}$$

here $L(p)$ is the logarithmic of $l(x_1,x_2,\_\_\_\_\_\_\_,x_n / p)$

$$h_k(p^*) = \left.\frac{\partial^k}{\partial p^k} h(p)\right|_{p=p^*}$$

$$\sigma^{*2} = -L_2^{-1}(p)\Big|_{p=p^*}$$

$$u(p^*) = \log\ g(p)\Big|_{p=p^*}$$

3. Loss function

Here we used two loss functions i.e. squared error loss function (SELF) and asymmetric precautionary loss function (APLF). The SELF is often used also because it is easy to compute and gives equal weightage to over estimation and under estimation. One can use some alternate loss function when the loss occurred is not symmetrical, such as asymmetric precautionary loss function.

3.1 Squared Error Loss Function (SELF)

A commonly used loss function is the squared error loss function (SELF)

$$L(p_B, p) = (p_B - p)^2 \tag{6}$$

3.2 Asymmetric Precautionary Loss Function (APLF)

A very useful and simple asymmetric precautionary loss function is

$$L(p_B, p) = \frac{(p_B - p)^2}{p_B} \tag{7}$$

4. Bayes estimators of p under different priors

If μ is given and k is fixed, then Bayes estimators of p may be obtained by using Lindley's approach under different priors by using the following steps.

First of all, we find find the likelihood function $l(x_1, x_2, \_\_\_\_\_\_\_, x_n / p)$ of the given probability density function $f(x/p)$ and the logarithm $L(p)$ of the above likelihood $l(x_1, x_2, \_\_\_\_\_\_\_, x_n / p)$. Then we find first two derivatives of $h(p)$ and three derivatives of $L(p)$ with respect to parameter p. After this, we find the M.L.E. $p*$ of p by solving the equation $L_1(p) = 0$. Then, σ^2 is obtained by using the formula $\sigma^2 = \left[-L_2(p)\right]^{-1}\Big|_{p=p*}$. In the next step, we find $u(p*)$ by using the formula $u(p*) = \log g(p)\big|_{p=p*}$,where $g(p)$ is the prior distribution of p. Lastly, we substitute all the values calculated above in equation (4) to find out Bayes estimator of $h(p)$ under the different priors.

If μ is given and k is fixed quantity, Bayes estimator of p may be obtained by using Lindley's approach by taking into consideration

$$h(p) = p, \quad h(p*) = p*, \quad h_1(p*) = 1, \quad h_2(p*) = 0$$

The likelihood function of (1) is

$$l(\underset{\wedge}{x}/\mu, p, k) = \frac{1}{p^n} \prod_{i=1}^{n} \left[1 + \frac{k(x_i - \mu)}{p}\right]^{-\left(\frac{1}{k}+1\right)} \tag{8}$$

$$L(p) = \log l(\underset{\wedge}{x}/\mu, p, k)$$

$$L(p) = -n \log p - \left(\frac{1}{k} + 1\right) \sum_{i=1}^{n} \log\left[1 + k\frac{(x_i - \mu)}{p}\right]$$

$$L_1(p) = -\frac{n}{p} + (1 + k) \sum_{i=1}^{n} \frac{(x_i - \mu)}{p\left[p + k(x_i - \mu)\right]} \tag{9}$$

$$L_2(p) = \frac{n}{p^2} - (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p + k(x_i-\mu)\right]}{p^2\left[p+k(x_i-\mu)\right]^2} \tag{10}$$

$$L_3(p) = -\frac{2n}{p^3} + 2(1+k)\sum_{i=1}^{n}\left[\frac{(x_i-\mu)\left[3p^2 + k^2(x_i-\mu)^2 + 3pk(x_i-\mu)\right]}{p^3[p+k(x_i-\mu)]^3}\right] \tag{11}$$

M.L.E. (p*) of p is obtained by solving

$$L_1(p) = 0$$

$$i.e. \qquad -\frac{n}{p} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)}{p\left[p+k(x_i-\mu)\right]} = 0$$

By using the error and trial method in the above equation, we get the maximum likelihood estimate p* of p .

$$\sigma^{*2} = \left[-L_2(p)\right]^{-1}$$

$$\sigma^{*2} = \frac{1}{-\frac{n}{p^2} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p+k(x_i-\mu)\right]}{p^2\left[p+k(x_i-\mu)\right]^2}} \tag{12}$$

$$Here \qquad h(p) = p$$

The Bayes estimator (p_B) of p given μ is given by

$$p_B = E\left[p/\underset{\wedge}{x}\right] \cong p^* + u_1(p^*)\,\sigma^{*2} + \frac{1}{2}\left[L_3(p^*)\right]\sigma^{*4}$$

The Bayes estimators (p_B) of p given μ under different priors are given below.

(i) Jeffrey's prior

Jeffrey's prior for the parameter p is given by

$$g(p) = \frac{1}{p} I_{(1,e)}(p)$$

$$Here \quad u(p^*) = \log \ g(p)\,|_{p=p^*}$$

$$= -\log p$$

$$u'(p) = -\frac{1}{p}$$

Hence the Bayes estimator (p_B^J) of p given μ by using (5) is

$$p_B^J = p^* + \left(-\frac{1}{p^*}\right)\left[\frac{1}{-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[2p^* + k(x_i-\mu)]}{p^{*2}\,[p^* + k(x_i-\mu)]^2}}\right]$$

$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^* k(x_i-\mu)\right]}{p^{*3}\,[p^* + k(x_i-\mu)]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[2p^* + k(x_i-\mu)]}{p^{*2}\,[p^* + k(x_i-\mu)]^2}\right]^2}$$

After simplifying the above equation, we get

$$p_B^J = p^* - \frac{1}{p^*}\left[\frac{1}{-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[2p^* + k(x_i-\mu)]}{p^{*2}\,[p^* + k(x_i-\mu)]^2}}\right]$$

$$+\frac{\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^* k(x_i-\mu)\right]}{p^{*3}\,[p^* + k(x_i-\mu)]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[2p^* + k(x_i-\mu)]}{p^{*2}\,[p^* + k(x_i-\mu)]^2}\right]^2} \quad (13)$$

(ii) Uniform Prior

Uniform prior for the parameter p is given by

$$g(p) = I_{(0,1)}(p)$$

$$\textit{Here}\quad u(p^*) = \log\ g(p)\,|_{p=p^*}$$

$$u(p) = \log 1 = 0$$

$$u'(p) = 0$$

Hence the Bayes estimator (p_B^U) of p given μ by using (5) is

$$p_B^U = p^* + (0)\left[\frac{1}{-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}}\right] + \frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^*k(x_i-\mu)\right]}{p^{*3}\left[p^* + k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$p_B^U = p^* + \frac{\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^*k(x_i-\mu)\right]}{p^{*3}\left[p^* + k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]^2} \tag{14}$$

(iii) Exponential prior

Exponential prior for the parameter p is given by

$$g(p) = e^{-p}; \qquad p > 0$$

$$Here \quad u(p^*) = \log\ g(p)\,|_{p=p^*}$$

$$u(p) = -p$$

$$u'(p) = -1$$

Hence the Bayes estimator (p_B^E) of p given μ by using (5) is

$$p_B^E = p^* + (-1)\left[\frac{1}{-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}}\right] + \frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^*k(x_i-\mu)\right]}{p^{*3}\left[p^* + k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$p_B^E = p* - \left[\frac{1}{-\frac{n}{p*^2} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p* + k(x_i-\mu)\right]}{p*^2\left[p* + k(x_i-\mu)\right]^2}} \right] + \frac{\left[-\frac{n}{p*^3} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[3p*^2 + k^2(x_i-\mu)^2 + 3p*k(x_i-\mu)\right]}{p*^3\left[p* + k(x_i-\mu)\right]^3} \right]}{\left[-\frac{n}{p*^2} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p* + k(x_i-\mu)\right]}{p*^2\left[p* + k(x_i-\mu)\right]^2} \right]^2} \tag{15}$$

(iv) Mukherjee-Islam Prior

Mukherjee-Islam prior for the parameter p is given by

$$g(p) = \frac{\alpha}{\sigma^\alpha} p^{\alpha-1}; \quad 0 < p < \sigma; \ \alpha > 0; \ \sigma > 0$$

$$Here \quad u(p*) = \log \ g(p)\,|_{p=p*}$$

$$u(p) = cons\tan t + (\alpha-1)\log p$$

$$u'(p) = \frac{(\alpha-1)}{p}$$

Hence the Bayes estimator (p_B^M) of p given μ by using (5) is

$$p_B^M = p* + \frac{(\alpha-1)}{p*}\left[\frac{1}{-\frac{n}{p*^2} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p* + k(x_i-\mu)\right]}{p*^2\left[p* + k(x_i-\mu)\right]^2}} \right] + \frac{\frac{1}{2}\left[-\frac{2n}{p*^3} + 2(1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[3p*^2 + k^2(x_i-\mu)^2 + 3p*k(x_i-\mu)\right]}{p*^3\left[p* + k(x_i-\mu)\right]^3} \right]}{\left[-\frac{n}{p*^2} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p* + k(x_i-\mu)\right]}{p*^2\left[p* + k(x_i-\mu)\right]^2} \right]^2}$$

After simplifying the above equation, we get

$$p_B^M = p^* + \left[\frac{(\alpha-1)/p^*}{-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}} \right] + \frac{\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^* k(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3} \right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2} \right]^2} \tag{16}$$

(v) Gamma prior

Gamma prior for the parameter p is given by

$$g(p) = \frac{1}{\sigma^\alpha \Gamma\alpha} p^{\alpha-1} e^{-p/\sigma}; \qquad \alpha, \sigma > 0; \quad p > 0$$

$$Here \quad u(p^*) = \log \ g(p)\,|_{p=p^*}$$

$$u(p) = constant + (\alpha-1)\log p - \frac{p}{\sigma}$$

$$u'(p) = \frac{(\alpha-1)}{p} - \frac{1}{\sigma}$$

Hence the Bayes estimator (p_B^G) of p given μ by using (5) is

$$p_B^G = p^* + \frac{\left[\frac{(\alpha-1)}{p^*} - \frac{1}{\sigma} \right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2} \right]} + \frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3p^* k(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3} \right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n} \frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2} \right]^2}$$

After simplifying the above equation, we get

$$p_B^G = p^* + \frac{\left(\frac{\alpha-1}{p^*} - \frac{1}{\sigma}\right)}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[2p^*+k(x_i-\mu)]}{p^{*2}[p^*+k(x_i-\mu)]^2}\right]} + \frac{\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[3p^{*2}+k^2(x_i-\mu)^2+3p^*k(x_i-\mu)]}{p^{*3}[p^*+k(x_i-\mu)]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)[2p^*+k(x_i-\mu)]}{p^{*2}[p^*+k(x_i-\mu)]^2}\right]^2} \tag{17}$$

4.1 Posterior Expected Losses under SELF

In this section, posterior expected losses of Bayes estimator (p_B) of p under SELF for different priors using Lindley's Approach are obtained.

Posterior expected loss of Bayes estimator (p_B) of p is $E(h(p)/\underset{\wedge}{x})$ where

$$E\left[h(p)/\underset{\wedge}{x}\right] = h(p^*) + \frac{1}{2}\left[h''(p^*) + 2h'(p^*)u'(p^*)\right]\sigma^{*2} + \frac{1}{2}\left[L_3(p^*)h'(p^*)\right]\sigma^{*4}$$

$$h(p) = (p_B - p)^2$$

$$h'(p) = -2(p_B - p)$$

$$h''(p) = 2$$

Here except $h(p)$ and its derivatives, all other quantities involved in equation (5) are same for all the corresponding priors as under section 4.

(i) Jeffrey's prior

Here

$$g(p) = \frac{1}{p} I_{(1,e)}(p)$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^J

$$E\left[h(p)/\underset{\wedge}{x}\right]_J = (p_B^J - p^*)^2 + \frac{\frac{1}{2}\left[2 + 2(-2)(p_B^J - p^*)\left(-\frac{1}{p^*}\right)\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]} + \frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)(p_B^J - p^*)}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_J = (p_B^J - p^*)^2 + \frac{\left[1+2\frac{(p_B^J - p^*)}{p^*}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$-\frac{2(p_B^J - p^*)\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{18}$$

(ii) Uniform prior

Here

$g(p) = I_{(0,1)}(p)$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^U

$$E\left[h(p)/\underset{\wedge}{x}\right]_U = (p_B^U - p^*)^2 + \frac{\frac{1}{2}\left[2+0\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)(p_B^U - p^*)}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_U = (p_B^U - p^*)^2 + \frac{1}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$-\frac{2(p_B^U - p^*)\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{19}$$

(iii) Exponential Prior

Here

$$g(p) = e^{-p}\ ; \qquad p > 0$$

(i) After substituting all the values in equation (5) we get the following expression of posterior expected l oss of p_B^E

$$E\left[h(p)/\underset{\wedge}{x}\right]_E = (p_B^E - p^*)^2 + \frac{\frac{1}{2}\left[2+2(-2)(p_B^E - p^*)(-1)\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$

$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)(p_B^E - p^*)}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_E = (p_B^E - p^*)^2 + \frac{\left[1+2(p_B^E - p^*)\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$

$$-\frac{2(p_B^E - p^*)\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \qquad (20)$$

(iv) Mukherjee-Islam prior

Here

$$g(p) = \frac{\alpha}{\sigma^{\alpha}}\, p^{\alpha - 1}\ ; \qquad 0 < p < \sigma;\ \ \alpha > 0;\ \ \sigma > 0$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^M

$$E\left[h(p)/\underset{\wedge}{x}\right]_{M-I}=(p_B^M-p^*)^2+\frac{\frac{1}{2}\left[2+2(-2)(p_B^M-p^*)\frac{(\alpha-1)}{p^*}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}}+2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)(p_B^M-p^*)}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_{M-I}=(p_B^M-p^*)^2+\frac{\left[1-2(\alpha-1)\frac{(p_B^M-p^*)}{p^*}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$-\frac{2(p_B^M-p^*)\left[-\frac{n}{p^{*3}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{21}$$

(v) Gamma prior

Here

$$g(p)=\frac{1}{\sigma^{\alpha}\overline{)\alpha}}p^{\alpha-1}e^{-p/\sigma}; \qquad \alpha,\sigma>0; \quad p>0$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^G

$$E\left[h(p)/\underset{\wedge}{x}\right]_{G}=(p_B^G-p^*)^2+\frac{\frac{1}{2}\left[2+2(-2)(p_B^G-p^*)\left(\frac{\alpha-1}{p^*}-\frac{1}{\sigma}\right)\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}}+2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3p^*k(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)(p_B^G-p^*)}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_G = (p_B^G - p^*)^2 + \frac{\left[1-2\left(\frac{\alpha-1}{p^*}-\frac{1}{\sigma}\right)(p_B^G-p^*)\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$

$$-\frac{2(p_B^G-p^*)\left[-\frac{n}{p^{*3}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{22}$$

4.2 Posterior Expected Loss under APLF

In this section, posterior expected losses of Bayes estimator (p_B) of p under APLF for different priors using Lindley's Approach are obtained.

Posterior expected loss of Bayes estimator (p_B) of p is $E(h(p)/\underset{\wedge}{x})$ where

$$h(p) = \frac{(p_B - p)^2}{p_B}$$

$$h'(p) = -\frac{2(p_B - p)}{p_B}$$

$$h''(p) = \frac{2}{p_B}$$

$$E\left[h(p)/\underset{\wedge}{x}\right] = h(p^*) + \frac{1}{2}\left[h''(p^*) + 2h'(p^*)u'(p^*)\right]\sigma^{*2} + \frac{1}{2}\left[L_3(p^*)h'(p^*)\right]\sigma^{*4}$$

Here except $h(p)$ and its derivative, all other quantities involved in equation (5) are same for all the corresponding priors as under section (5.1).

(i) Jeffrey's prior

Here

$$g(p) = \frac{1}{p} I_{(1,e)}(p)$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^J

$$E\left[h(p)/\underset{\wedge}{x}\right]_J^1 = \frac{(p_B^J-p^*)^2}{p_B^J} + \frac{\frac{1}{2}\left[\frac{2}{p_B^J}+2(-2)\frac{(p_B^J-p^*)}{p_B^J}\left(-\frac{1}{p^*}\right)\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}}+2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk\left(x_i-\mu\right)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)\frac{(p_B^J-p^*)}{p_B^J}}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_J^1 = \frac{(p_B^J-p^*)^2}{p_B^J} + \frac{\left[\frac{1}{p_B^J}+2\frac{(p_B^J-p^*)}{p_B^J p^*}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$-\frac{2\frac{(p_B^J-p^*)}{p_B^J}\left[-\frac{n}{p^{*3}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk\left(x_i-\mu\right)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{23}$$

(ii) Uniform prior

Here

$$g(p) = I_{(0,1)}(p)$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^U

$$E\left[h(p)/\underset{\wedge}{x}\right]_U^1=\frac{(p_B^U-p^*)^2}{p_B^U}+\frac{\frac{1}{2}\left[\frac{2}{p_B^U}+0\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}}+2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk\left(x_i-\mu\right)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)\frac{(p_B^U-p^*)}{p_B^U}}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_U^1=\frac{(p_B^U-p^*)^2}{p_B^U}+\frac{1/p_B^U}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$-\frac{2\frac{(p_B^U-p^*)}{p_B^U}\left[-\frac{n}{p^{*3}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk\left(x_i-\mu\right)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{24}$$

(iii) Exponential Prior

Here

$$g(p)=e^{-p}\,; \qquad p>0$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^E

$$E\left[h(p)/\underset{\wedge}{x}\right]_E^1=\frac{(p_B^E-p^*)^2}{p_B^E}+\frac{\frac{1}{2}\left[\frac{2}{p_B}+2(-2)\frac{(p_B^E-p^*)}{p_B^E}(-1)\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}}+2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk\left(x_i-\mu\right)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)\frac{(p_B^E-p^*)}{p_B^E}}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k\left(x_i-\mu\right)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_E^1 = \frac{(p_B^E - p^*)^2}{p_B^E} + \frac{\left[\frac{1}{p_B^E} + 2\frac{(p_B^E - p^*)}{p_B^E}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]}$$
$$- \frac{2\frac{(p_B^E - p^*)}{p_B^E}\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3pk(x_i-\mu)\right]}{p^{*3}\left[p^* + k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]^2} \quad (25)$$

(iv) Mukherjee-Islam prior

Here

$$g(p) = \frac{\alpha}{\sigma^{\alpha}} p^{\alpha-1}; \quad 0 < p < \sigma; \;\; \alpha > 0; \;\; \sigma > 0$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^M

$$E\left[h(p)/\underset{\wedge}{x}\right]_{M-I}^1 = \frac{(p_B^M - p^*)^2}{p_B^M} + \frac{\frac{1}{2}\left[\frac{2}{p_B^M} + 2(-2)\frac{(p_B^M - p^*)}{p_B^M}\frac{(\alpha-1)}{p^*}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]}$$
$$+ \frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}} + 2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3pk(x_i-\mu)\right]}{p^{*3}\left[p^* + k(x_i-\mu)\right]^3}\right]*(-2)\frac{(p_B^M - p^*)}{p_B^M}}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_{M-I}^1 = \frac{(p_B^M - p^*)^2}{p_B^M} + \frac{\left[\frac{1}{p_B^M} - \frac{2(\alpha-1)}{p^*}\frac{(p_B^M - p^*)}{p_B^M}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]}$$
$$- \frac{2\frac{(p_B^M - p^*)}{p_B^M}\left[-\frac{n}{p^{*3}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2} + k^2(x_i-\mu)^2 + 3pk(x_i-\mu)\right]}{p^{*3}\left[p^* + k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}} + (1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^* + k(x_i-\mu)\right]}{p^{*2}\left[p^* + k(x_i-\mu)\right]^2}\right]^2} \quad (26)$$

(v) Gamma Prior

Here

$$g(p)=\frac{1}{\sigma^{\alpha}\Gamma\alpha}p^{\alpha-1}e^{-p/\sigma};\qquad \alpha,\sigma>0;\quad p>0$$

After substituting all the values in equation (5) we get the following expression of posterior expected loss of p_B^G

$$E\left[h(p)/\underset{\wedge}{x}\right]_G^1=\frac{(p_B^G-p^*)^2}{p_B^G}+\frac{\frac{1}{2}\left[\frac{2}{p_B^G}+2(-2)\frac{(p_B^G-p^*)}{p_B^G}\left(\frac{\alpha-1}{p^*}-\frac{1}{\sigma}\right)\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$+\frac{\frac{1}{2}\left[-\frac{2n}{p^{*3}}+2(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3p^*k(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]*(-2)\frac{(p_B^G-p^*)}{p_B^G}}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2}$$

After simplifying the above equation, we get

$$E\left[h(p)/\underset{\wedge}{x}\right]_G^1=\frac{(p_B^G-p^*)^2}{p_B^G}+\frac{\left[\frac{1}{p_B^G}-2\left(\frac{\alpha-1}{p^*}-\frac{1}{\sigma}\right)\frac{(p_B^G-p^*)}{p_B^G}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]}$$
$$-\frac{2\frac{(p_B^G-p^*)}{p_B^G}\left[-\frac{n}{p^{*3}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[3p^{*2}+k^2(x_i-\mu)^2+3pk(x_i-\mu)\right]}{p^{*3}\left[p^*+k(x_i-\mu)\right]^3}\right]}{\left[-\frac{n}{p^{*2}}+(1+k)\sum_{i=1}^{n}\frac{(x_i-\mu)\left[2p^*+k(x_i-\mu)\right]}{p^{*2}\left[p^*+k(x_i-\mu)\right]^2}\right]^2} \tag{27}$$

(5) Numerical Illustration

To illustrate the calculations of Bayes estimates of scale parameter (p) and their posterior expected losses under SELF and APLF with known location parameter (μ) and fixed shape parameter (k) under different priors, one real data for Generalized Pareto distribution have been used and the results obtained are given in Table 1. Besides, we have generated two random samples each of size 100 from Generalized Pareto model with the help of Easy Fit Professional software 5.0. The Bayes estimates of p for different priors along with their posterior expected losses under SELF and APLF are obtained for the generated samples and are given in Tables 2and 3 respectively.

5.1 Real life application of generalized pareto distribution (k<0)

The following is the graphical representation (Histogram) of a real life data of bacterial leaf blight disease observed in barley crop. In this study, the percentage of bacterial leaf blight disease on 57 barley plants was observed for a week. The data were observed to follow generalized Pareto distribution. This was done by means of EasyFit software. Goodness of fit was tested by Kolmogorov Smirnov and Anderson Darling tests. The graphical representation of the data is given below.

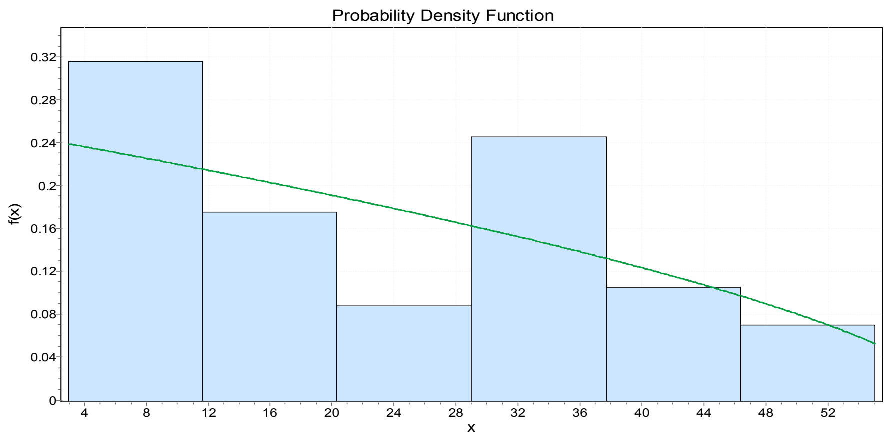

Figure 1: Probability curve of generalized Pareto model (p=35.787, k=-0.60972, μ=1.7683)

We have obtained Bayes estimates of the fitted data under different priors. The posterior expected losses for different priors under Squared Error Loss Function (SELF) and Asymmetric Precautionary Loss Function (APLF) have also been obtained. The abovesaid values are given in Table 1.

Prior	p_B	**Posterior expected loss under SELF**	**Posterior expected loss under APLF**
Jeffrey's	35.437233	1.494450	0.042172
Uniform	35.507926	1.352222	0.038082
Exponential	33.071385	0.489860	0.014812
Mukh.-Islam α=1	35.507926	1.352222	0.038082
α=2	35.578619	1.199999	0.033728
α=3	35.649312	1.037780	0.029111
Gamma α=1 σ=1	33.071385	0.489860	0.014812
α=2 σ=1	33.142077	0.682129	0.020582
α=3 σ=1	33.212770	0.864403	0.026026

Table 1: Bayes estimates of the parameter p and the corresponding values of posterior expected losses under SELF and APLF for real data

Priors	p_B	**Posterior expected loss under SELF**	**Posterior expected loss under APLF**
Uniform	7.865266	0.254359	0.0323396
Jeffrey's	7.830557	0.261503	0.033395
Exponential	7.596442	0.246754	0.032482
Mukherjee-Islam α = 1	7.865266	0.254359	0.032339
α = 2	7.899975	0.244806	0.030988
α = 3	7.934685	0.232843	0.029345
Gamma α = 1 σ=1	7.596442	0.246754	0.032482
α = 2 σ=1	7.631152	0.255862	0.033528
α = 3 σ=1	7.665861	0.262560	0.034250

Table 2: Bayes estimates of the parameter p and the values of posterior expected losses under SELF and APLF (p=8, μ=5, k= -0.35) for generated sample I

Priors	p_B	**Posterior expected loss under SELF**	**Posterior expected loss under APLF**
Uniform	44.50002	1.556025	0.034966
Jeffrey's	44.45462	1.613800	0.036302
Exponential	42.50969	0.217937	0.005126
Mukherjee-Islam α = 1	44.50002	1.556025	0.034966
α = 2	44.54541	1.494127	0.033541
α = 3	44.59081	1.428106	0.032026
Gamma α = 1 σ=1	42.50969	0.217937	0.005126
α = 2 σ=1	42.55509	0.336756	0.007913
α = 3 σ=1	42.60049	0.451452	0.010597

Table 3: Bayes estimates of the parameter p and the values of posterior expected losses under SELF and APLF (p=45, μ=25, k= -0.5) for generated sample II

6. Comparison and Conclusion

The results are compared by calculating the values of posterior expected losses under SELF and APLF using the estimates of p under different priors. It is revealed from the Tables 1, 2 and 3 respectively that the posterior expected loss corresponding to Asymmetric Precautionary Loss Function (APLF) is less compared to Squared Error Loss Function (SELF) under all the priors for real as well as generated samples. Thus, it is concluded that APLF is superior to SELF for obtaining Bayes estimate of p given the values of μ and k. Further, the value of posterior expected loss is minimum under Exponential prior or gamma prior with α=1, σ=1 (which coincides with exponential prior) for real data set and generated dataset II, whereas it is minimum under Mukherjee-Islam prior with α=1 for generated dataset I. Moreover, under Mukherjee-Islam prior as α increases, the estimated value of p approaches to its true value and the corresponding loss also decreases. Therefore it may be appropriate to use Mukherjee-Islam prior with a large value of α to estimate the scale parameter p under APLF.

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