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A NEW LIFETIME DISTRIBUTION FOR MODELING MONOTONIC DECREASING SURVIVAL PATTERNS

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Abstract

One parameter inverse size biased p-dimensional Rayleigh distribution (ISBRD) is introduced for modeling lifetimes. Its distributional properties including moments are studied. Hazard function is studied under different parametric settings. Parameter estimation is done using maximum likelihood method and Bayesian approach. The risks of Bayes estimators have been obtained under different loss structures and a comparative risk analysis has been conducted empirically. Findings of simulation study are presented.

Key Words- Squared Error, Linex, Entropy and Precautionary Loss, Maximum Likelihood Estimates, Asymptotically Invariant Prior, Posterior Risk functions.

1. Introduction

The Rayleigh distribution (Rayleigh, 1919) is frequently used as a model for the analysis of data resulting from investigation involving wind velocity, wave propagation, radiation and target error by physicists and engineers. The p-dimensional Rayleigh distribution was introduced by Cohen and Whitten (1988). In the present article, we propose inverse size-biased p-dimensional Rayleigh distribution. Earliest mention of the notion of weighted distributions is found in Fisher (1934). Length or size biased sampling introduced by Cox (1962) is an example of weighted distribution. Multiple illustrations on the concept of size biased distributions are found in Patil (2002). These ideas have found wide applicability in disease mapping, survival and determination intermediate (latency) period, the size bias implies that a unit with a large value of the variable has a greater chance of being selected. For instance, let us consider two variables (for investigation of spread of a specific disease)-area of a region (say, village) and the number of infected individuals under each area. Then the arithmetic mean of the represent proportion of regions with i infected people. This represents average from the region's viewpoint. However, proportion of infected people in the i person region represents average from the viewpoint of the infected person. The latter describes size biased average category size. Thus, size biased sampling and modeling is useful for a human population exposed to a contagious microorganism such as Zika or HINI. Such length biased sampling and modeling mechanism can also be applied to the cases of children reported with dyslexia where the disorder exists in different degrees among the affected children and could be corrected through effective and timely intervention.

The probability density function of one parameter inverse size biased p-dimensional Rayleigh distribution (ISBRD) is obtained as

$$f(y;\alpha)=\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right);\ y>0,\alpha>0,p>0 \tag{1.1}$$

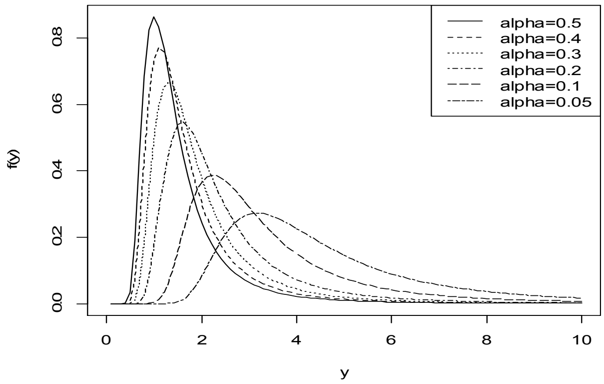

Figure 1: Probability Density function of ISBRD

The probability that an individual survives longer than time t, is called survival function. As t ranges from 0 to ∞, the monotone survival curve goes to 0. The survival function is a lower incomplete Gamma function.

$$\overline{F}(t)=\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right);t>0 \tag{1.2}$$

The hazard function represents the conditional failure time. It exhibits higher probability of meeting mortality in the beginning of lifetime for the proposed ISBRD and takes the following form

$$h(t)=\frac{2}{\alpha^{(p+1)/2}}\frac{1}{t^{p+2}}\exp\left(-\frac{1}{\alpha t^2}\right)\left(\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)\right)^{-1} \tag{1.3}$$

The Graphs of survival and hazard rate functions under different parametric settings of α is shown in figure 2 and figure 3.

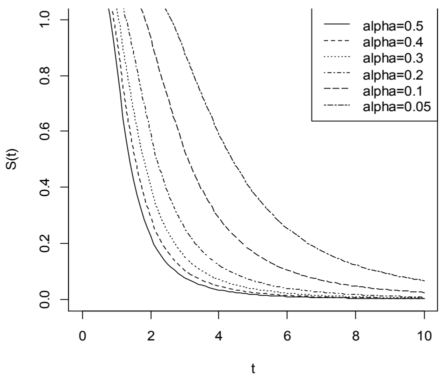

Figure 2: Survival function of ISBRD

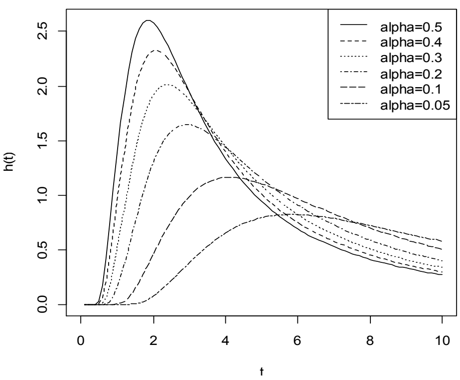

Figure 3: Survival function of ISBRD

The survival function is a monotonic decreasing function while the hazard function increases initially during infancy or early lifetime then declines at a decreasing pace. This reveals that an ISB p-dim Rayleigh distribution is suitable to model lifetimes of the items which have a higher chance of failing during early lifetime, but after survival to a specific development level, the probability of failure decreases as time increases.

The present paper considers Bayesian estimation of the scale parameter α under four types of loss functions. The first is squared error loss (quadratic loss) function (SELF), which is classified as a symmetric function and associates equal importance to the losses due to overestimation and underestimation of equal magnitude and is measured as

$$L(\alpha,\hat{\alpha}) = (\alpha - \hat{\alpha})^2 \tag{1.4}$$

However, symmetrical loss functions are not appropriate when overestimation and underestimation are not equally serious. The convex loss function known as linear exponential loss function (LINEX) which is an asymmetric loss function for a parameter α, was introduced by Varian (1975) as

$$L(\Delta) = \left(e^{k\delta} - k\delta - 1\right) \; ; \quad k \neq 0 \tag{1.5}$$

Where, $\delta = \dfrac{\hat{\alpha}}{\alpha} - 1$ .This asymmetric loss function has been found to be appropriate in the situations where either overestimation is more serious than underestimation and vice-versa. The positive value of *k* is used when overestimation is more serious than underestimation and for negative value of *k,* reverse is true. For *k* close to zero, this loss function is approximately squared error loss and therefore symmetric. These loss functions have been studied by several authors, among them Canfield (1970), Zellner (1986), Rojo (1987), Basu and Ebrahimi (1991), Pandey and Rai (1992) and Soliman (2000).

By using non-symmetric loss functions one is able to deal with the cases where it is more damaging to miss the target on one side than the other. The General Entropy loss function (GELF) was developed by Calabria and Pulcini (1994) as

$$L(\delta) = \left(\delta^d - d\log_e(\delta) - 1\right); \; \delta = \frac{\hat{\alpha}}{\alpha} \; and \; d = 1 \tag{1.6}$$

Introduced by Norstrom (1996), precautionary loss function is asymmetric in nature, which has the quadratic loss-function as a special case. These loss functions approach infinity near the origin to prevent underestimates and thus give conservative estimates, especially when, for example, low failure rates are being estimated. The conservative estimates make these loss functions useful when the consequences are major and under-estimation is serious. When dealing with disastrous consequences such as disease modeling, it can be worse to underestimate the potentiality of an event than to overestimate it. It has the following form

$$L(\hat{\alpha},\alpha) = \frac{(\hat{\alpha} - \alpha)^2}{\hat{\alpha}} \tag{1.7}$$

The rest of the paper is organized as follows. Some statistical characteristics for ISB p-dim Rayleigh distribution are derived under section two. Parameter estimation under classical and Bayesian approaches and posterior risk analysis under different loss functions, for the complete sample case, is undertaken in section three. Empirical investigation of various risk estimators is carried out in section four. Section five summarizes the contribution of the present paper.

2. Distributional Properties

The r[th] order raw moment is given by

$$\mu_r' = \frac{\Gamma\left(\frac{p-(r-1)}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}$$

The r[th] order Central moments is given by

$$\mu_r = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\left(\frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}\right)^i\left(\frac{\Gamma\left(\frac{p-(r-i-1)}{2}\right)}{\alpha^{(r-i)/2}\Gamma\left(\frac{p+1}{2}\right)^r}\right)$$

Now the mean and variance are given by

$$E(y) = \frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}$$

$$V(y) = \frac{1}{\alpha\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^2\right)$$

Median is $\frac{1}{3}\left(\left(\frac{2}{\alpha(p+2)}\right)^{1/2}+2\frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\right)$

Mode is unique and given by $\left(\frac{2}{\alpha(p+2)}\right)^{1/2}$ $for\ \alpha > 0$.

Standard Deviation is $\frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^2\right)^{1/2}$

Mean Deviation is $M.D = \frac{4}{5}\frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^2\right)^{1/2}$

Quartile Deviation is $Q.D=\frac{4}{6}\frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^{2}\right)^{1/2}$

Coefficient of Variation is given by

$$C.V=\frac{100}{\Gamma\left(\frac{p}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^{2}\right)^{1/2}$$

Harmonic mean is written as $H=\frac{\Gamma\left(\frac{P+1}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{P+2}{2}\right)}$

Geometric mean is derived as $\log G=\frac{\alpha^{p/2}\Gamma\left(\frac{p}{2}\right)}{(\log\alpha)^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}$

The above distribution properties are empirically evaluated for p=2 band varying sample size to study the pattern of changes with changing α, which is the scale parameter.

A	Mean	Var	Med.	Mode	S.D.	M.D.	Q.D.	C.V.	H.M.	G.M.
1	1.128	0.726	1.085	0.707	0.961	0.769	0.641	75.551	0.886	23.623
1.5	0.921	0.484	0.886	0.577	0.785	0.628	0.523	75.551	0.723	22.905
2	0.797	0.363	0.767	0.5	0.680	0.544	0.453	75.551	0.626	13.663
2.5	0.713	0.290	0.686	0.447	0.608	0.486	0.405	75.551	0.560	11.237
3	0.651	0.242	0.626	0.408	0.555	0.444	0.370	75.551	0.511	10.271
3.5	0.603	0.207	0.580	0.377	0.514	0.411	0.342	75.551	0.473	9.841
4	0.564	0.181	0.542	0.353	0.480	0.384	0.320	75.551	0.443	9.661
4.5	0.531	0.161	0.511	0.333	0.453	0.362	0.302	75.551	0.417	9.618
5	0.504	0.145	0.485	0.316	0.430	0.344	0.286	75.551	0.396	9.654
5.5	0.481	0.132	0.462	0.301	0.410	0.328	0.273	75.551	0.377	9.742

Table 1: Some statistical features of ISBRD for p=2.

Var→Variance, Med→Median, S.D.→Standard Deviation, M.D.→Mean Deviation, Q.D.→Quartile Deviation, C.V.→Covariance Variation, H.M.→Harmonic Mean, G.M.→Geometric Mean

3. Parameter Estimation

If n items are put to test, then the joint likelihood function for complete sample is given by

$$l(y;\alpha)=\frac{2^n}{\alpha^{n(p+1)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^n}\prod_{i=1}^{n}\left(\frac{1}{y_i^{p+2}}\right)\exp\left(-\sum_{i=1}^{n}\left(\frac{1}{\alpha y_i^2}\right)\right) \tag{3.1}$$

The maximum likelihood estimate (MLE) of the scale parameter is $\hat{a}=\frac{2A}{n(p+1)}$ for $A=\sum_{i=1}^{n}\left(\frac{1}{y_i^2}\right)$. Assuming no information is available, as may happen with a new contagion like Zika, asymptotically invariant prior, proposed by Hartigan (1964) which is of the form $g(\alpha)=\frac{1}{\alpha^3},\alpha>0$ is adopted for the posterior analysis. In conjunction with the likelihood (3.1) it yields the following posterior density function of ISB p-dim. Rayleigh distribution

$$\Pi\left(\alpha\middle|\underline{y}\right)=\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}} \tag{3.2}$$

The following posterior analysis is carried out assuming that p is known:

Theorem 1: For a positive integer p and α > 0, under SELF, Bayes estimator of α is the posterior mean

$$\hat{\alpha}_{BS}=\frac{A\Gamma\left(\frac{n(p+1)+6}{2}\right)}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}=\frac{2A}{n(p+1)+6}$$

The posterior risk function of $\hat{\alpha}_{BS}$, under SELF is

$$R_{BS}\left(\hat{\alpha}_{BS}\right)=\alpha^2\left\{\frac{2\alpha^2(n(p+1)+8)(n(p+1)+10)}{(n(p+1))(n(p+1)+6)}\left(\frac{(n(p+1)+12)}{(n(p+1)+6)}-1\right)+1\right\} \tag{3.3}$$

Theorem 2: For a positive integer p and k > 0, under LINEX loss function, Bayes estimator of α is

$$\hat{\alpha}_{BL}=\frac{A}{k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]$$

The posterior risk function of $\hat{\alpha}_{BL}$, under SELF is

$$R_{BS}\left(\hat{\alpha}_{BL}\right)=\alpha^2\left[\frac{\alpha^2(n(p+1)+8)(n(p+1)+10)}{k(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\left(\left\{\frac{(n(p+1)+12)}{4k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\right\}-2\right)+1\right] \tag{3.4}$$

Theorem 3: For a positive integer p and α > 0, under GELF, Bayes estimator of α is

$$\hat{\alpha}_{BE}=\frac{A\Gamma\left(\frac{n(p+1)+8}{2}\right)}{\Gamma\left(\frac{n(p+1)+10}{2}\right)}=\frac{2A}{(n(p+1)+8)}$$

The posterior risk function of $\hat{\alpha}_{BE}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BE})=\alpha^2\left\{\frac{8\alpha^2(n(p+1)+10)}{(n(p+1))(n(p+1)+8)}+1\right\}$$

$$R_{BS}(\hat{\alpha}_{BE})=\frac{2A}{(n(p+1)+8)}\left[\frac{2A}{(n(p+1)+8)}-2\alpha\right]+\alpha^2 \quad (3.5)$$

Theorem 4: For a positive integer p and α > 0, under Precautionary loss, Bayes estimator of α is

$$\hat{\alpha}_{BP}=\frac{A\left[\Gamma\left(\frac{n(p+1)+4}{2}\right)\right]^{1/2}}{\left[\Gamma\left(\frac{n(p+1)+8}{2}\right)\right]^{1/2}}=\frac{2A}{[(n(p+1)+4)(n(p+1)+6)]^{1/2}} \quad (3.6)$$

The posterior risk function of $\hat{\alpha}_{BP}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BP})=\alpha^2\left[\frac{2\alpha^2(n(p+1)+8)(n(p+1)+10)}{n(p+1)}\left\{\frac{(n(p+1)+12)}{(n(p+1)+4)(n(p+1)+6)}-\frac{1}{[(n(p+1)+4)(n(p+1)+6)]^{1/2}}\right\}+1\right] \quad (3.7)$$

Posterior risk is found to be a function of the sample data and of the prior parameters.

4. Risk Analysis

Comparison in terms of risks identifies the decision rule with the lowest risk. A decision rule is admissible (with respect to the loss function) if and only if no other rule dominates it, otherwise it is inadmissible. A comparison of this type maybe needed to check whether an estimator is inadmissible under some loss function. If it is so, then the estimator would not be used for the losses specified by that loss function. For this purpose risk of the estimators relative to squared error loss have been estimated. It is evident from the expressions of the risks of the estimators that an analytical comparison of these risks is not possible. Therefore, an empirical comparison is made. Random samples of size n=6 have been generated from (1) for α=0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55. We compute and report the corresponding risks, for complete sample case, so as to observe and compare the behavior of the following risk functions for the proposed lifetime distribution:

$$R_{BS}(\hat{\alpha}_{BS}), R_{BS}(\hat{\alpha}_{BL}), R_{BS}(\hat{\alpha}_{BE}), R_{BS}(\hat{\alpha}_{BP})$$

α	RS	RL	RE	RP
0.1	0.0101	0.0108	0.0100	0.0103
0.15	0.0229	0.0265	0.0227	0.0245
0.2	0.0413	0.0527	0.0407	0.0462
0.25	0.0657	0.0937	0.0644	0.0775
0.3	0.0968	0.1547	0.0938	0.1212
0.35	0.1351	0.2425	0.1296	0.1802
0.4	0.1815	0.3647	0.1723	0.2584
0.45	0.2371	0.5305	0.2221	0.3602
0.5	0.3026	0.7499	0.2799	0.4903
0.55	0.3796	1.0345	0.3463	0.6543
0.6	0.4692	1.3967	0.4220	0.8583
0.65	0.5729	1.8504	0.5079	1.1089
0.7	0.6923	2.4106	0.6049	1.4133
0.75	0.8291	3.0934	0.7139	1.7792
0.8	0.9851	3.9164	0.8360	2.2151
0.85	1.1623	4.8981	0.9723	2.7298
0.9	1.3628	6.0582	1.1240	3.3329
0.95	1.5887	7.4178	1.2923	4.0346
1	1.8425	8.9992	1.4786	4.8454

Table 2: Posterior risk functions at p=2 and n=6

RS→risk function of SELF, RL→risk function of LINEX loss function,
RE→risk function of general entropy, RP→risk function of precautionary.

The following is the graph of posterior risk functions under the SELF, LINEX, general entropy and precautionary loss functions on p=2 and n=6.

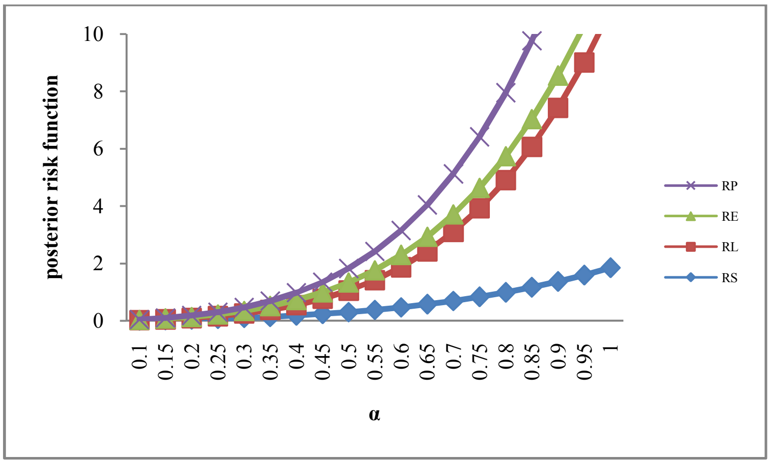

Figure 4: Posterior risk function of ISBRD

5. Conclusion

A new lifetime distribution is proposed to alternatively model lifetime of units which age rapidly or wherein the infection spreads rapidly initially and is quickly arrested under treatment but to different degrees, sometimes resulting in no recovery at all. The theoretical properties of the introduced distribution are derived. Risk analysis is discussed within a Bayes framework. In risk analysis, both the potentiality of an undesired event and its consequences are investigated in terms of the performance of estimators assessed on the basis of their relative posterior risk which is found to be the least under GELF (Table 2). Therefore, the corresponding Bayes estimator is regarded as the most preferred. It is also observed that the Bayes estimators dominate when the asymmetric loss functions are used and therefore are deemed more appropriate, instead of the quadratic loss function for proposed life time distribution.

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Appendix A

1. The proof of probability density function:

The probability density function of the p-dimensional Rayleigh distribution is

$$f(x;\sigma)=\frac{2^{-(p-2)/2}}{\sigma\Gamma\left(\frac{p}{2}\right)}\left(\frac{x}{\sigma}\right)^{p-1}\exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2}\right),\ x>0,\sigma>0 \qquad \text{(i)}$$

Putting the value of $\alpha = 2\sigma^2$ in equation (i), then we will get

$$f(x;\alpha)=\frac{2}{\Gamma\left(\frac{p}{2}\right)}\frac{x^{p-1}}{\alpha^{p/2}}\exp\left(-\frac{x^2}{\alpha}\right),x>0,\alpha>0 \qquad \text{(ii)}$$

Where, α is a scale parameter.

The mean of p-dimensional Rayleigh distribution is

$$E(x)=\alpha^{1/2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}$$

Then the probability density function of size biased p-dimensional Rayleigh distribution is

$$g(x;\alpha)=\frac{xf(x;\alpha)}{E(x)}$$

$$f(x;\alpha)=\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}x^{p}\exp\left(-\frac{x^2}{\alpha}\right),x>0,\alpha>0$$

And now the Inverse size biased p-dimensional Rayleigh distribution is given by

$$f(y;\alpha)=\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right),y>0,\alpha>0$$

Where, y=1/x.

2. The proof of Survival function:

$$\overline{F}(t)=\int_t^\infty f(y;\alpha)dy$$

$$\overline{F}(t)=\int_t^\infty \frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right)dy$$

$$\overline{F}(t)=\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\int_0^{\frac{1}{\alpha t^2}} z^{((p+1)/2)-1}e^{-z}dz\text{ , where } z=\frac{1}{\alpha y^2}$$

$$\overline{F}(t)=\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)$$

This is the lower incomplete function.

3. The proof of Hazard rate function:
We have,

$$h(t)=\frac{f(t;\alpha)}{\overline{F}(t)}$$

$$h(t)=\frac{\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right)}{\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)}$$

$$h(t)=\frac{2}{\alpha^{(p+1)/2}}\frac{1}{t^{p+2}}\exp\left(-\frac{1}{\alpha t^2}\right)\left(\gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right)\right)^{-1}$$

Appendix B

The proof of distribution properties
The $\mathbf{r^{th}}$ **order raw moment** is given by

$$E\left(y^r\right)=\int_0^\infty y^r\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right)dy$$

$$E\left(y^r\right)=\frac{1}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}\int_0^\infty k^{(p-r-1)/2}e^{-k}dk\text{ , Where, }k=\frac{1}{\alpha y^2}$$

$$\mu_r^{'}=E\left(y^r\right)=\frac{\Gamma\left(\frac{p-(r-1)}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)}$$

Central moments:

$$\mu_r=E(y-E(x))^r$$

$$\mu_r=\int_0^\infty (y-E(x))^r f(y,\alpha)dy$$

$$\mu_r = \int_0^\infty \left(y - \frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)} \right)^r \frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right) dy$$

Let, $\beta = \dfrac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}$, then

$$\mu_r = \int_0^\infty (y-\beta)^r \frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right) dy$$

Where, $(y-\beta)^r = {}^rC_0 y^r(-\beta)^0 - {}^rC_1 y^{r-1}(-\beta)^1 + {}^rC_2 y^{r-2}(-\beta)^2$

$$\mu_r = \int_0^\infty {}^rC_0 y^r(-\beta)^0 - {}^rC_1 y^{r-1}(-\beta)^1 + {}^rC_2 y^{r-2}(-\beta)^2$$

$$\mu_r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(\frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{r/2}\Gamma\left(\frac{p+1}{2}\right)} \right)^i \frac{\Gamma\left(\frac{p-(r-i+1)}{2}\right)}{\alpha^{(r-i)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^r}$$

Median:

Median=1/3(mode+2mean)

$$median = \frac{1}{3}\left(\left(\frac{2}{\alpha(p+2)}\right)^{1/2} + 2\frac{\Gamma\left(\frac{p}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)} \right)$$

Mode:

$$\frac{d}{dy}\left[\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{y^{p+2}} \exp\left(-\frac{1}{\alpha y^2}\right) \right] = 0$$

$$y = \left(\frac{2}{\alpha(p+2)}\right)^{1/2}, \alpha > 0$$

Standard Deviation:

S.D = (variance)$^{1/2}$

$$S.D = \frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)} \left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right) - \left(\Gamma\left(\frac{p}{2}\right)\right)^2 \right)^{1/2}$$

Mean Deviation:

$$\frac{5}{6}M.D=\frac{2}{3}S.D$$

$$M.D=\frac{4}{5}\frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^{2}\right)^{1/2}$$

Quartile Deviation:

$$Q.D=\frac{4}{6}M.D$$

$$Q.D=\frac{4}{6}\frac{1}{\alpha^{1/2}\Gamma\left(\frac{p+1}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^{2}\right)^{1/2}$$

Coefficient of Variation:

$CV=\frac{S.D.}{mean}\times 100$ **,** where, S.D=standard deviation

$$C.V=\frac{100}{\Gamma\left(\frac{p}{2}\right)}\left(\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)-\left(\Gamma\left(\frac{p}{2}\right)\right)^{2}\right)^{1/2}$$

Harmonic Mean:

$$\frac{1}{H}=\int_{0}^{\infty}\frac{1}{y}\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^{2}}\right)dy$$

$\frac{1}{H}=\frac{\alpha^{1/2}}{\Gamma\left(\frac{p+1}{2}\right)}\int_{0}^{\infty}z^{p/2}e^{-z}dz$, where $z=\frac{1}{\alpha y^{2}}$

$$H=\frac{\Gamma\left(\frac{P+1}{2}\right)}{\alpha^{1/2}\Gamma\left(\frac{P+2}{2}\right)}$$

Geometric Mean:

$$\log G=\int_{0}^{\infty}\log y\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^{2}}\right)dy$$

$$\log G=\frac{\alpha^{p/2}\Gamma\left(\frac{p}{2}\right)}{(\log\alpha)^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}$$

Appendix C

The proof of posterior density function:
The likelihood function is given by

$$l(y;\alpha)=\frac{2^n}{\alpha^{n(p+1)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}\prod_{i=1}^{n}\left(\frac{1}{y_i^{p+2}}\right)\exp\left(-\sum_{i=1}^{n}\left(\frac{1}{\alpha y_i^2}\right)\right)$$

The posterior density function is

$$\Pi\left(\alpha\middle|\underline{y}\right)=\frac{l(y;\alpha)g(\alpha)}{\int_0^\infty l(y;\alpha)g(\alpha)d\alpha}\text{, where } g(\alpha)=\frac{1}{\alpha^3},\ \alpha>0$$

$$\Pi\left(\alpha\middle|\underline{y}\right)=\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}$$

And $$f(y;\hat{\alpha})=\frac{1}{\alpha^{n(p+1)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+2}{2}\right)}\left(\frac{n(p+1)\hat{\alpha}}{2}\right)^{(n(p+1)+2)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)$$

where, the maximum likelihood function (MLE) is $\hat{\alpha}=\frac{2A}{n(p+1)}$

Proof of theorem 1: The square error loss function (SELF)

$$\hat{\alpha}_{BS}=\int_0^\infty \alpha\frac{1}{\alpha^{(n(p+1)+6)/2}}\frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}A^{(n(p+1)+8)/2}e^{-\frac{A}{\alpha}}d\alpha$$

$$\hat{\alpha}_{BS}=\frac{A\Gamma\left(\frac{n(p+1)+6}{2}\right)}{\Gamma\left(\frac{n(p+1)+8}{2}\right)}=\frac{2A}{n(p+1)+6}$$

The posterior risk function of $\hat{\alpha}_{BS}$, under SELF is

$$R_{BS}\left(\hat{\alpha}_{BS}\right)=E_\alpha\left(\hat{\alpha}_{BS}^2\right)-2\alpha E_\alpha\left(\hat{\alpha}_{BS}\right)+\alpha^2$$

$$E_\alpha\left(\hat{\alpha}_{BS}^2\right)=\frac{4\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+6)^2\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+12)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha\left(\hat{\alpha}_{BS}^2\right)=\frac{2\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+6)^2}$$

$$E_\alpha(\hat{\alpha}_{BS}) = \frac{2\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+6)\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+10)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BS}) = \frac{\alpha^3(n(p+1)+10)(n(p+1)+8)}{(n(p+1))(n(p+1)+6)}$$

$$R_{BS}(\hat{\alpha}_{BS}) = \frac{2\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+6)^2} - 2\alpha\frac{\alpha^3(n(p+1)+8)(n(p+1)+10)}{(n(p+1))(n(p+1)+6)} + \alpha^2$$

$$R_{BS}(\hat{\alpha}_{BS}) = \alpha^2\left\{\frac{2\alpha^2(n(p+1)+8)(n(p+1)+10)}{(n(p+1))(n(p+1)+6)}\left(\frac{(n(p+1)+12)}{(n(p+1)+6)}-1\right)+1\right\}$$

Proof of theorem 2:

The Linear Exponential loss function (LINEX),

If $\delta = \frac{\hat{\alpha}}{\alpha} - 1$, then the LINEX loss function $L(\delta)$

$$L(\delta) = \left(e^{k\delta} - k\delta - 1\right);\quad k \neq 0$$

$$L(\alpha,\hat{\alpha}) = \left\{\exp\left(k\left(\frac{\hat{\alpha}}{\alpha}-1\right)\right) - k\left(\frac{\hat{\alpha}}{\alpha}-1\right) - 1\right\} = e^{-k}\left\{\exp\left(k\left(\frac{\hat{\alpha}}{\alpha}\right)\right) - k\left(\frac{\hat{\alpha}}{\alpha}\right) + k - 1\right\}$$

The Bayes estimation $\hat{\sigma}$ of σ under the Linex loss function is the solution of

$$\frac{d}{d\alpha}E\{L(\alpha,\hat{\alpha})\} = 0$$

$$E\{L(\alpha,\hat{\alpha})\} = e^k\left\{E\left[\exp\left(k\left(\frac{\hat{\alpha}}{\alpha}\right)\right)\right] - kE\left(\frac{\hat{\alpha}}{\alpha}\right) + k - 1\right\}$$

Writing $\hat{\alpha}$ as $\hat{\alpha}_{BL}$ we have,

$$\frac{d}{d\hat{\alpha}_{BL}}L(\alpha,\hat{\alpha}) = E_p\left\{e^{-k}\frac{\hat{\alpha}_{BL}}{\alpha}\exp\left(k\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)\right)\right\} - E_p\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right) = 0$$

That $\hat{\alpha}_{BL}$ is the solution to the following equation

$$E_p\left\{\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)e^{k\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)}\right\} = e^k E_p\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)$$

The integrating on both sides with respect to α, we get

$$\int_0^\infty \left(\frac{\hat{\alpha}_{BL}}{\alpha}\right) e^{k\left(\frac{\hat{\alpha}_{BL}}{\alpha}\right)} \frac{1}{\alpha^{(n(p+1)+6)/2}} \frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)} A^{(n(p+1)+8)/2} e^{-\frac{A}{\alpha}} d\alpha$$

$$= e^k \int_0^\infty \left(\frac{\hat{\alpha}_{BL}}{\alpha}\right) \frac{1}{\alpha^{(n(p+1)+6)/2}} \frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)} A^{(n(p+1)+8)/2} e^{-\frac{A}{\alpha}} d\alpha$$

$$\int_0^\infty \frac{1}{\alpha^{(n(p+1)+8)/2}} \exp\left\{-\left(\frac{A-k\hat{\alpha}_{BL}}{\alpha}\right)\right\} d\alpha = e^k \int_0^\infty \frac{1}{\alpha^{(n(p+1)+8)/2}} e^{-\frac{A}{\alpha}} d\alpha$$

$$\hat{\alpha}_{BL} = \frac{A}{k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]$$

The posterior risk function of $\hat{\alpha}_{BL}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BL}) = E_\alpha(\hat{\alpha}_{BL}^2) - 2\alpha E_\alpha(\hat{\alpha}_{BL}) + \alpha^2$$

$$E_\alpha(\hat{\alpha}_{BL}^2) = \frac{\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]^2}{k^2\alpha^{(n(p+1)+6)/2}\Gamma\left(\frac{n(p+1)+8}{2}\right)} \int_0^\infty \hat{\alpha}^{(n(p+1)+12)/2} \exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right) d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BL}^2) = \frac{\alpha^4(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{4k^2(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]^2$$

$$E_\alpha(\hat{\alpha}_{BL}) = \frac{\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]}{k\alpha^{(n(p+1)+6)/2}\Gamma\left(\frac{n(p+1)+8}{2}\right)} \int_0^\infty \hat{\alpha}^{(n(p+1)+10)/2} \exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right) d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BL}) = \frac{\alpha^3(n(p+1)+10)(n(p+1)+8)}{k(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]$$

$$R_{BS}(\hat{\alpha}_{BL}) = \alpha^2\left[\frac{\alpha^2(n(p+1)+8)(n(p+1)+10)}{k(n(p+1))}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\left(\left\{\frac{(n(p+1)+12)}{4k}\left[1-\exp\left(\frac{2k}{n(p+1)+10}\right)\right]\right\}-2\right)+1\right]$$

Proof of theorem 3:

For a positive integer p and α > 0, under GELF, Bayes estimator of α is

$$\hat{\alpha}_{BE} = \left(E_\alpha\left(\frac{1}{\alpha}\right)\right)^{-1}$$

$$\hat{\alpha}_{BE} = \left(\int_0^\infty \frac{1}{\alpha} \frac{1}{\alpha^{(n(p+1)+6)/2}} \frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)} A^{(n(p+1)+8)/2} e^{-\frac{A}{\alpha}} d\alpha\right)^{-1}$$

$$\hat{\alpha}_{BE} = \frac{A\Gamma\left(\frac{n(p+1)+8}{2}\right)}{\Gamma\left(\frac{n(p+1)+10}{2}\right)} = \frac{2A}{(n(p+1)+8)}$$

The posterior risk function of $\hat{\alpha}_{BE}$, under SELF is

$$R_{BS}(\hat{\alpha}_{BE}) = E_\alpha(\hat{\alpha}^2_{BE}) - 2\alpha E_\alpha(\hat{\alpha}_{BE}) + \alpha^2$$

$$E_\alpha(\hat{\alpha}^2_{BE}) = \frac{4\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+8)^2\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+12)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}^2_{BE}) = \frac{2\alpha^4(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+8)}$$

$$E_\alpha(\hat{\alpha}_{BE}) = \frac{2\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+8)\Gamma\left(\frac{n(p+1)+8}{2}\right)}\int_0^\infty \hat{\alpha}^{(n(p+1)+10)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_\alpha(\hat{\alpha}_{BE}) = \frac{\alpha^3(n(p+1)+10)}{(n(p+1))}$$

$$R_{BS}(\hat{\alpha}_{BE}) = \frac{2\alpha^4(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+8)} - 2\frac{\alpha^4(n(p+1)+10)}{(n(p+1))} + \alpha^2$$

$$R_{BS}(\hat{\alpha}_{BE}) = \alpha^2\left\{\frac{8\alpha^2(n(p+1)+10)}{(n(p+1))(n(p+1)+8)} + 1\right\}$$

Proof of theorem 4:

For a positive integer p and α > 0, under Precautionary loss, Bayes estimator of α is

$$\hat{\alpha}_{BE} = \left(E_\alpha(\alpha^2)\right)^{1/2}$$

$$\hat{\alpha}_{BE} = \left(\int_0^\infty \alpha^2 \frac{1}{\alpha^{(n(p+1)+6)/2}} \frac{1}{\Gamma\left(\frac{n(p+1)+8}{2}\right)} A^{(n(p+1)+8)/2} e^{-\frac{A}{\alpha}} d\alpha\right)^{1/2}$$

$$\hat{\alpha}_{BP} = \frac{A\left[\Gamma\left(\frac{n(p+1)+4}{2}\right)\right]^{1/2}}{\left[\Gamma\left(\frac{n(p+1)+8}{2}\right)\right]^{1/2}} = \frac{2A}{\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}$$

The posterior risk function of $\hat{\alpha}_{BP}$, under SELF is

$$R_{BS}\left(\hat{\alpha}_{BP}\right)=E_{\alpha}\left(\hat{\alpha}_{BP}^{2}\right)-2\alpha E_{\alpha}\left(\hat{\alpha}_{BP}\right)+\alpha^{2}$$

$$E_{\alpha}\left(\hat{\alpha}_{BP}^{2}\right)=\frac{4\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+12)/2}}{\alpha^{(n(p+1)+6)/2}(n(p+1)+4)(n(p+1)+6)\Gamma\left(\frac{n(p+1)+8}{2}\right)}$$

$$\int_{0}^{\infty}\hat{\alpha}^{(n(p+1)+12)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_{\alpha}\left(\hat{\alpha}_{BP}^{2}\right)=\frac{2\alpha^{4}(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+4)(n(p+1)+6)}$$

$$E_{\alpha}\left(\hat{\alpha}_{BP}\right)=\frac{2\left(\frac{n(p+1)}{2}\right)^{(n(p+1)+10)/2}}{\alpha^{(n(p+1)+6)/2}\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}\Gamma\left(\frac{n(p+1)+8}{2}\right)}$$

$$\int_{0}^{\infty}\hat{\alpha}^{(n(p+1)+10)/2}\exp\left(-\frac{n(p+1)\hat{\alpha}}{2\alpha}\right)d\hat{\alpha}$$

$$E_{\alpha}\left(\hat{\alpha}_{BP}\right)=\frac{\alpha^{3}(n(p+1)+10)(n(p+1)+8)}{(n(p+1))\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}$$

$$R_{BS}\left(\hat{\alpha}_{BP}\right)=\frac{2\alpha^{4}(n(p+1)+8)(n(p+1)+10)(n(p+1)+12)}{(n(p+1))(n(p+1)+4)(n(p+1)+6)}$$

$$-2\frac{\alpha^{4}(n(p+1)+10)(n(p+1)+8)}{(n(p+1))\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}+\alpha^{2}$$

$$R_{BS}\left(\hat{\alpha}_{BP}\right)=\alpha^{2}\left[\frac{2\alpha^{2}(n(p+1)+8)(n(p+1)+10)}{n(p+1)}\left\{\frac{(n(p+1)+12)}{(n(p+1)+4)(n(p+1)+6)}\right.\right.$$

$$\left.\left.-\frac{1}{\left[(n(p+1)+4)(n(p+1)+6)\right]^{1/2}}\right\}+1\right]$$