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STATISTICAL PROPERTIES AND APPLICATIONS OF THE EXPONENTIATED INVERSE KUMARASWAMY DISTRIBUTION

Kawsar Fatima, Uzma Jan and S. P. Ahmad
Department of Statistics, University of Kashmir, Srinagar, India
E Mail: kawsarfatima@gmail.com; peeruzma1@gmail.com; sprvz@yahoo.com

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Abstract

This manuscript proposes the new Exponentiated Inverse Kumaraswamy distribution. We have studied some statistical properties of this distribution. The maximum likelihood procedure is employed to calculate the parameters. At last, two real life problems are used for illustration which confirm that the proposed model can be used well in ascertaining real data.

Key Words: Inverse Kumaraswamy, Exponentiated Inverse Kumaraswamy distribution, Statistical properties and Maximum Likelihood Estimation.

1. Introduction

Poondi Kumaraswamy (1980) derived the Kumaraswamy distribution. It can be applied to several real life problems whose outcomes have lower and upper bounds. It has been identified and discussed by Jones (2009) in detail.

The two-parameter Inverse Kumaraswamy (IKum) distribution otherwise known as the Inverted Kumaraswamy distribution was introduced by AL-Fattah et al (2017). Inverse Kumaraswamy (IKum) distribution as a lifetime model and it has been widely used in applied fields. The pdf and cdf of the IKD are given as

$$g(x;\alpha,\beta)=\alpha\beta(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta-1} \quad ;0<x<\infty\ ,\ \alpha,\beta>0 \qquad (1.1)$$

$$G(x;\alpha,\beta)=(1-(1+x)^{-\alpha})^{\beta} \quad ;0<x<\infty\ ,\quad \alpha,\beta>0\ , \qquad (1.2)$$

where α and β are two shape parameters.

The concept of Exponentiated distributions have been considered commonly in statistics such as Mudholkar et al (1995) established the Exponentiated Weibull distribution. Lemonte and Cordeiro (2011) developed the exponentiated generalized inverse Gaussian distribution. Flaih et al. (2012) postulated the exponentiated inverted Weibull (EIW) distribution. Fatima and Ahmad (2017) studied the Exponentiated inverse Exponential Distribution while Fatima et al (2017) introduced a generalization of Minimax distribution.

2. Exponentiated Inverse Kumaraswamy Distribution

The Exponentiated distributions are derived by adding a positive real number to the cdf of an arbitrary parent distribution to the power say $\lambda > 0$. Its pdf is given by

$$f(x) = \lambda\, g(x)\left[G(x)\right]^{\lambda-1} . \tag{2.1}$$

As such, the corresponding cdf is given by

$$F(x) = \left[G(x)\right]^{\lambda} ,\ \lambda > 0 \tag{2.2}$$

With this understanding, we put Equations (1.1) and (1.2) into Equation (2.1) to get the pdf of the EIKD as

$$f(x) = \alpha\beta\lambda(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1};\ 0 < x < \infty\ ,\ \alpha,\beta,\lambda > 0. \tag{2.3}$$

Then, the corresponding cdf of the Exponentiated inverse Kumaraswamy distribution is given by

$$F(x) = \left[1-(1+x)^{-\alpha}\right]^{\beta\lambda}\ ;\ 0 < x < \infty\ ,\ \ \beta,\alpha,\lambda > 0, \tag{2.4}$$

with α , β and λ being shape parameters

The plots of pdf and cdf are depicted for various values of parameters α , β and λ have been presented in Figures 1and 2 respectively.

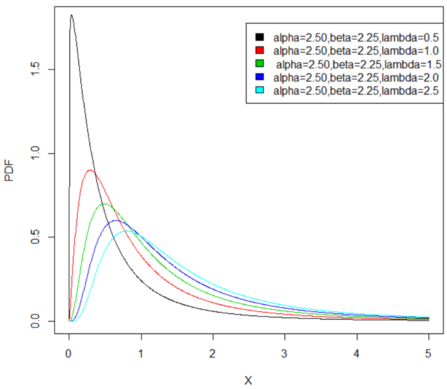

Figure 1 The graph of density function

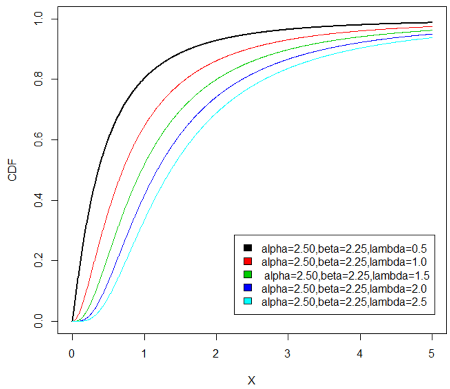

Figure 2 The graph of distribution function

Fig. 1 illustrates the shapes of EIKD. Fig. 1 shows that the pdf of EIKD is unimodal and with skewness towards right and becomes more peaked with decreasing the value of λ .

3. Special Cases

a) If we put $\lambda = 1$, then EIKD (2.3) reduce IKD.

b) If we put $\lambda = \beta = 1$, then EIKD (2.3) reduces to Lomax distribution (LD).

c) If we put $\alpha = \lambda = 1$, then EIKD (2.3) reduces to inverted beta Type II distribution (IBD)

d) If we put $\alpha = \beta = 1$, then EIKD (2.3) reduces to inverted beta Type II distribution

e) If we put $\alpha = 1$, then EIKD (2.3) reduces to two parameter inverted beta Type II distribution.

f) If we put $\alpha=\beta=\lambda=1$, then Exponentiated inverse Kumaraswamy distribution (2.3) reduces to log-logistic (Fisk) distribution.

4. Reliability Analysis

This sub-section considers with presenting the survival analysis of the proposed Exponentiated inverse Kumaraswamy distribution.

4.1 Reliability function

It is given by

$$R(x)=1-\{(1-(1+x)^{-\alpha})\}^{\beta\lambda} \tag{4.1}$$

4.2 Hazard Rate

It is given by

$$H(x)=\frac{\alpha\beta\lambda(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1}}{1-\{(1-(1+x)^{-\alpha})\}^{\beta\lambda}}\ ; \quad 0<x<\infty, \alpha,\beta,\lambda>0. \tag{4.2}$$

The plots for the equations (4.1) and (4.2) are displayed in Figures 3 and 4 respectively.

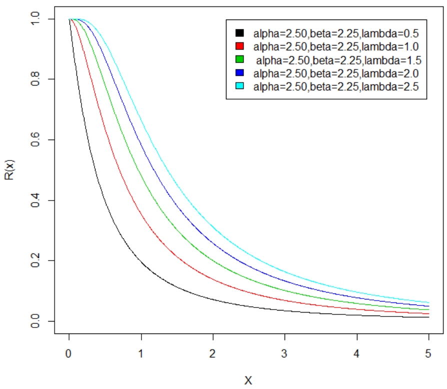

Figure 3 The graph of Reliability function

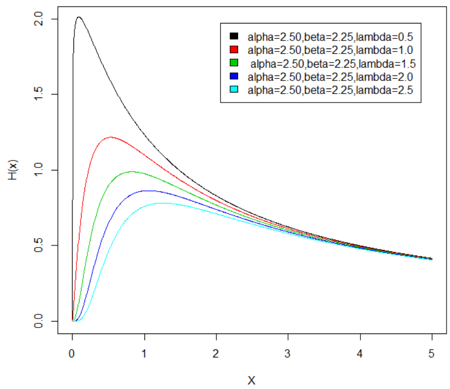

Figure 4 The graph of Hazard function

Fig. 4 indicates that the shape of the hazard rate is unimodal, it increases at the initial stage and later decreases. We can also say that the hazard rate shows an inverted bathtub shape.

4.3 Reverse Hazard function $\phi(x)$

The reverse hazard function $\phi(x)$ is defined to be

$$\phi(x)=\alpha\beta\lambda(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{-1};\ 0<x<\infty, \alpha,\beta,\lambda>0. \tag{4.3}$$

5. Structural Properties of Exponentiated inverse Kumaraswamy Distribution

Theorem 5.1 Let $X=(X_1,X_2,\ldots,X_n)$ be a sample from the EIKD with pdf

$f(x)=\alpha\beta\lambda\,(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1};\ 0<x<\infty\ ,\ \alpha,\beta,\lambda>0.$

Then

$$E(X^r)=\lambda\beta\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\,B\left(1-\frac{j}{\alpha},\beta\lambda\right)\qquad r=1,2,\dots .$$

Proof: Since we know that the rth moment of a r.v x is given by

$$E(X^r)=\int_0^\infty X^r f(x)\,dx\quad . \tag{5.1}$$

Now using eq. (2.3) in eq. (5.1), we have

$$E(X^r)=\int_0^\infty X^r\alpha\beta\lambda\,(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1}dx\,. \tag{5.2}$$

Put $\quad (1+x)^{-\alpha}=t$; as $x\to 0,t\to 1,$; as $x\to\infty,t\to 0,$

We apply the series expansion

$$\left(t^{\frac{-1}{\alpha}}-1\right)^r=\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\,t^{\frac{-j}{\alpha}}$$

Using the above equation in (5.2) we get

$$E(X^r)=\lambda\beta\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\int_0^1 t^{\left(1-\frac{j}{\alpha}\right)-1}(1-t)^{\lambda\beta-1}dt\,.$$

On solving the above equation, we get

$$E(X^r)=\lambda\beta\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\,B\left(1-\frac{j}{\alpha},\beta\lambda\right),r=1,2,\dots,\ \alpha>j,\ j=0,1,2...r \tag{5.3}$$

For r=1 in (5.3), the expected value of EIKD is given by

$$\mu_1'=\lambda\beta B\left(1-\frac{1}{\alpha},\beta\lambda\right),\qquad \alpha>1. \tag{5.4}$$

If we put r=2 in eq. (5.3), we have

$$\mu_2'=\lambda\beta B\left(1-\frac{2}{\alpha},\beta\lambda\right),\qquad \alpha>2. \tag{5.5}$$

Thus, the variance of Exponentiated inverse Kumaraswamy distribution is given by

$$\mu_2=\beta\lambda B\left(1-\frac{2}{\alpha},\beta\lambda\right)-\left\{\beta\lambda B\left(1-\frac{1}{\alpha},\beta\lambda\right)\right\}^2\ . \tag{5.6}$$

6. The Mode of the Exponentiated Inverse Kumaraswamy Distribution

Taking the logarithm of (2.3) as follows:

$$\log f(x)=\log(\alpha\beta\lambda)-(\alpha+1)\log(1+x)+(\beta\lambda-1)\log(1-(1+x)^{-\alpha}). \tag{6.1}$$

For mode, we have

$$\frac{\partial}{\partial x}\log f(x) = \frac{-(\alpha+1)}{(1+x)} + \frac{(\beta\lambda-1)}{(1-(1+x)^{-\alpha})}\alpha(1+x)^{-(\alpha+1)} \quad (6.2)$$

Equating (6.2) to zero, we get

$$x_0 = \left[\left(\frac{\alpha+1}{\alpha\beta\lambda+1}\right)^{-1/\alpha} - 1\right]. \quad (6.3)$$

7. Quantile Function and Median

The quantiles of EIKD is as follows

$$Q(u) = \left[\left(1-u^{\frac{1}{\beta\lambda}}\right)^{-1/\alpha} - 1\right] \quad , \quad (7.1)$$

where u ~U (0,1). For u=1/2, the median is as:

$$\text{Median} = F^{-1} = \left[\left(1-0.5^{\frac{1}{\beta\lambda}}\right)^{-1/\alpha} - 1\right] \quad . \quad (7.2)$$

8. Moment generating function and Characteristic function

Theorem 8.1 Let X have an Exponentiated IKum distribution. Then mgf of X is given by:

$$M_X(t) = E(e^{tx}) = \lambda\beta \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{t^k}{k!} \frac{(-1)^j \Gamma(r+1)}{\Gamma(r+1-j)j!} B\left(\beta\lambda, 1-\frac{j}{\alpha}\right) \quad (8.1)$$

Proof: By definition

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f(x)dx$$

$$\Rightarrow \qquad M_X(t) = \lambda\beta \sum_{r=0}^{\infty}\sum_{j=0}^{r} \frac{t^r}{r!}\binom{r}{j}(-1)^{r-j} B\left(1-\frac{j}{\alpha}, \beta\lambda\right).$$

This completes the proof.

Theorem 8.2 Let X have an Exponentiated IKum distribution. Then characteristic function of X is:

$$\phi_X(t) = E(e^{itx}) = \lambda\beta \sum_{r=0}^{\infty}\sum_{j=0}^{r} \frac{(it)^r}{r!}\binom{r}{j}(-1)^{r-j} B\left(1-\frac{j}{\alpha}, \beta\lambda\right) \quad (8.2)$$

Proof: By definition

$$\phi_X(t) = E(e^{itx}) = \int_0^\infty e^{itx} f(x)dx$$

$$= \int_0^\infty \left(1 + itx + \frac{(itx)^2}{2!} + \cdots \right) f(x) dx$$

$$\Rightarrow \quad \phi_X(t) = \lambda\beta \sum_{r=0}^{\infty} \sum_{j=0}^{r} \frac{(it)^r}{r!} \binom{r}{j} (-1)^{r-j} B\left(1 - \frac{j}{\alpha}, \beta\lambda\right)$$

This completes the proof.

9. Shannon's entropy of EIKD

For EIKD, it is obtained as:

$$H(x) = -E[\log f(x)] = -E[\log\{\alpha\beta\lambda\,(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1}\}]$$

$$H(x) = -\log[\alpha\beta\lambda] - (\alpha+1)I_1 - (\beta\lambda - 1)I_2 \quad (9.1)$$

Now, $I_1 = E(-\log(1+x)) = \int_0^\infty -\log(1+x)\{\alpha\beta\lambda\,(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1}\}dx$

$$I_1 = \beta\lambda \int_0^\infty \log(1+x)^{-\alpha}(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1} dx \quad (9.2)$$

$$I_1 = \frac{\beta\lambda}{\alpha} B(1, \beta\lambda)(\psi(1) - \psi(1+\beta\lambda)) = \frac{\beta\lambda}{\alpha} \frac{\Gamma(\beta\lambda)\Gamma 1}{\Gamma(\beta\lambda+1)}(\psi(1) - \psi(1+\beta\lambda))$$

$$\Rightarrow \quad I_1 = \frac{1}{\alpha}(\psi(1) - \psi(1+\beta\lambda)) \quad (9.3)$$

Also,

$$I_2 = E\left(\log(1-(1+x)^{-\alpha}\right) = \int_0^\infty \log(1-(1+x)^{-\alpha})\{\alpha\beta\lambda(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1}\}dx$$

$$I_2 = \alpha\beta\lambda \int_0^\infty \log(1-(1+x)^{-\alpha})(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta\lambda-1} dx \quad (9.4)$$

$$\Rightarrow \quad I_2 = \beta\lambda \int_0^1 y^{\beta\lambda-1} y^{1-1} \log y dy \quad (9.5)$$

On solving the equation (9.5), we get

$$I_2 = (\psi(\beta\lambda) - \psi(\beta\lambda+1)) \quad (9.6)$$

Substitute the values of (9.3) and (9.6) in equation (9.1)

$$H(x) = \frac{(\alpha+1)}{\alpha}(\psi(1+\beta\lambda) - \psi(1)) + (\beta\lambda - 1)(\psi(\beta\lambda+1) - \psi(\beta\lambda)) - \log[\alpha\beta\lambda] \quad (9.7)$$

10. Order Statistics

The pdf of the k^{th} order statistic $X_{(k)}$ from EIKD iss obtained as:

$$f_X(k) = \frac{n!}{(k-1)!(n-k)!} \alpha\lambda\beta(1+x)^{-(\alpha+1)}\left\{1-(1+x)^{-\alpha}\right\}^{\beta\lambda k-1}[1-\{1-(1+x)^{-\alpha}\}^{\beta\lambda}]^{n-k} \quad (10.1)$$

For $k = n$, the pdf $X_{(n)}$ is therefore:

$$f_X(n) = n\alpha\lambda\beta(1+x)^{-(\alpha+1)}\left\{1-(1+x)^{-\alpha}\right\}^{n\beta\lambda-1}$$

For $k = 1$, the pdf of $X_{(1)}$ is given by:

$$f_X(1) = n\alpha\lambda\beta(1+x)^{-(\alpha+1)}\left\{1-(1+x)^{-\alpha}\right\}^{\beta\lambda-1}[1-\{1-(1+x)^{-\alpha}\}^{\beta\lambda}]^{n-1}$$

11. Estimation of Parameters

In a view to approximate the parameters of the EIKD, the procedure of MLE is considered. Assume a random sample of size n with pdf (2.3) and likelihood function as

$$L(\underline{x}) = \alpha^n \beta^n \lambda^n \prod_{i=1}^{n} (1+x_i)^{-(\alpha+1)} (1-(1+x_i)^{-\alpha})^{\beta\lambda-1} \tag{11.1}$$

Applying log, we get

$$\ln L(\underline{x}) = n\ln\alpha + n\ln\beta + n\ln\lambda - (\alpha+1)\sum_{i=1}^{n}\ln(1+x_i) + (\beta\lambda-1)\sum_{i=1}^{n}\ln(1-(1+x_i)^{-\alpha}) \tag{11.2}$$

Differentiating (11.2) with respect to each of the parameters we get

$$\frac{\partial \ln L(\underline{x})}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log(1+x_i) + \alpha(\beta\lambda-1)\sum_{i=1}^{n}\left(\frac{(1+x_i)^{-(\alpha+1)}}{(1-(1+x_i)^{-\alpha})}\right) \tag{11.3}$$

$$\frac{\partial \ln L(\underline{x})}{\partial\beta} = \frac{n}{\beta} + \lambda\sum_{i=1}^{n}\ln(1-(1+x_i)^{-\alpha}) \tag{11.4}$$

$$\frac{\partial \ln L(\underline{x})}{\partial\lambda} = \frac{n}{\lambda} + \beta\sum_{i=1}^{n}\ln(1-(1+x_i)^{-\alpha})\,. \tag{11.5}$$

Equating (11.3), (11.4) and (11.5) to zero and simultaneously solving them provides the values of α, β and λ using software like R.

12. Application

This segment deals with illustrating the usefulness of the Exponentiated Inverse Kumaraswamy distribution. We fit this model to lifetime problems and compare the outcome with its special cases that are, Inverse Kumaraswamy distribution, Lomax distribution, Inverted Beta Type II distribution, and two parameter Inverted Beta Type II distributions. The first problem gives the monthly actual taxes revenue in Egypt (in 1,000 million Egyptian pounds) between January 2006 and November 2010 given in Nassar and Nada (2011). Recently, Oguntunde et al., (2016) studied these data using Lindley Exponential distribution.

The second data set represents the waiting times (in minutes) before service of 100 Bank customers. The data has been previously used by Ghitany *et al* (2008) and Owoloko et al., (2016).

The necessary numerical assessment is carried out using R software.

Distribution	Parameter Estimates			-2Log L	AIC	AICC	BIC
	α	β	λ				
EIKD	1.78734 (0.16349)	7.82065 (2.71667)	8.01124 (2.78288)	386.9007	392.9007	392.4434	399.1333
IKD	1.21808 (0.10333)	14.43764 (3.34505)	–	414.6139	418.6139	418.8282	422.769
LD	0.39199 (0.05103)	–	–	529.534	531.534	531.6042	533.6115
IBD	–	10.95233 (1.42587)	–	426.8383	428.8383	428.9085	430.9158
IBD(2par.)	–	3.225718 (762.6108)	3.395315 (802.7062)	426.8383	430.8383	431.0526	434.9934

Table 1: MLEs (S.E in parentheses) with information criteria on tax revenue

Distribution	Parameter Estimates			-2Log L	AIC	AICC	BIC
	α	β	λ				
EIKD	1.41982 (0.11395)	3.61765 (2.81214)	3.55660 (2.76468)	657.5584	663.5584	663.8084	671.3739
IKD	1.167724 (0.09307)	7.468873 (1.314922)	–	665.1152	669.1152	669.2389	674.3255
LD	0.458433 (0.04843)	–	–	792.2565	794.2565	794.2973	796.8617
IBD	–	6.241195 (0.624119)	–	674.2574	676.2574	676.2982	678.8626
IBD(2par.)	–	2.494983 (321.8166)	2.501498 (322.6569)	674.2574	678.2574	678.3811	683.4677

Table 2: MLEs (S.E in parentheses) with information criteria on waiting time of bank customers

For comparison, the criterion like AIC, AICC and BIC are taken into consideration. The distribution with less AIC, AICC and BIC values is regarded as better.

From Table 1 and 2, it is obvious that the EIKD provides lesser values of AIC, AICC and BIC than other models. As such it can be concluded that the Exponentiated Inverse Kumaraswamy distribution provides better fit than Inverse Kumaraswamy, Lomax, inverted Beta Type II and two parameter inverted Beta Type II distributions.

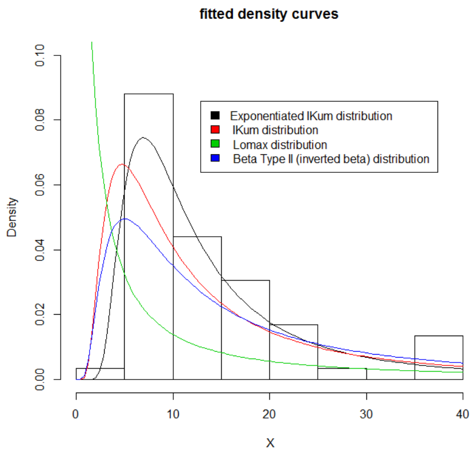

Figure 5 Fitted densities for data on tax revenue

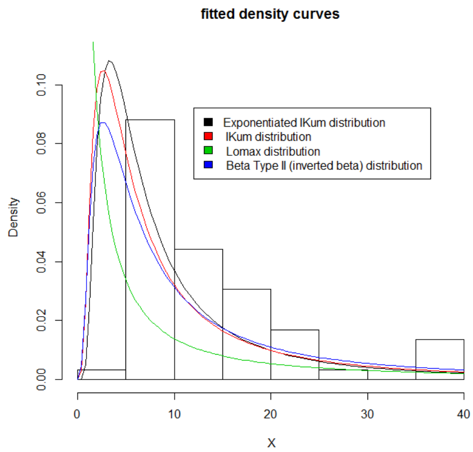

Figure 6 Fitted densities for data on waiting time of bank customers

7. Conclusion

In this paper, we have introduced Exponentiated Inverse Kumaraswamy distribution, which acts as a generalization to so many distributions viz. Inverse Kumaraswamy, Lomax, inverted Beta Type II, and two parameter inverted Beta Type II distributions. After introducing Exponentiated Inverse Kumaraswamy distribution, we have investigated its several statistical properties. Two real data sets have been considered in order to make comparison between special cases of Exponentiated Inverse Kumaraswamy distribution in terms of fitting. After the fitting of Exponentiated Inverse Kumaraswamy distribution and its special cases to the data sets considered, the results are given in Table 1and 2. It is clear from the Table 1 and 2 that, Exponentiated Inverse Kumaraswamy distribution possesses minimum values of AIC, AICC and BIC on its fitting, to two real life data sets. Therefore, we can conclude that the Exponentiated Inverse Kumaraswamy distribution will be treated as a best fitted distribution to the data sets as compared to its other special cases.

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